EXERCISE BOOK IN ALGEBRA

A REVISION OF McCURDY'S
EXERCISE BOOK IN ALGEBRA

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PREFACE

Thirty years ago the late Matthew S. McCurdy published his "Exercise Book in Algebra" which was "intended primarily to be supplementary to some regular text-book," but which he "hoped would be found useful as an independent review and drill book." That it amply fulfilled the author's purpose is indicated by the valuable service of the book through many school generations. That Mr. McCurdy wrought with wisdom and skill is attested by the popularity of his book through this long period.

The many changes in school courses and in college entrance requirements, however, have made imperative a thorough revision. Although following Mr. McCurdy's original plan, the writer has attempted to adapt the book to present requirements. The result is a book which is in some respects entirely new, but which retains many exercises from the original edition.

The following are the outstanding changes:

1. Rearrangement of exercises and topics.

2. Elimination of several topics such as permutations and combinations, undetermined coefficients, and continued fractions.

3. Elimination of numerous exercises which were too difficult.

4. The introduction of a large number of simple exercises, many of which are adapted for oral work and all of which should be helpful to pupils who are reviewing the subject as well as to beginners.

5. A new chapter on simple equations and problems has been inserted early in the work.

6. The material on exponents and radicals has been rearranged and the number of exercises increased.

7. New chapters have been added as follows: theory of quadratics, graphs, logarithms, and numerical trigonometry.

8. A great many problems have been added under the various types of equations, and some problems have been eliminated.

9. The groups of review exercises have been increased in number and in scope, making cumulative reviews a feature of the book.

10. Many exercises from college entrance examinations have been used, and a group of College Entrance Examination Board examinations in "Elementary Algebra Complete" replaces the miscellaneous college examinations of the previous edition.

In its present form the book fully meets the college requirements in elementary algebra and contains ample material for pupils who are preparing for college entrance examinations.

CONTENTS

EXERCISE BOOK IN ALGEBRA

I. NUMERICAL SUBSTITUTION

A

1.

When $a = 8$, $b = -4$, $c = 2$, find the value of:

1. $a + b + c$.
2. $a + b - c$.
3. $a - b - c$.
4. $ac - b$.
5. $a - bc$
6. $\frac{a + b}{c}$.
7. $\frac{ac}{b}$.
8. $ab - ac + bc$.
9. $2a - c^2$.
10. $a^2 - bc$.
11. $\frac{c^2 - b}{a}$.
12. $a + bc$.
13. $\frac{a + bc}{c}$.
14. $3a + 2b - 7c$.
15. $a - b^2 + 3c^2$.

B

If $a = 8$, $b = 1$, $c = 2$, $d = 3$, $x = 4$, $y = 5$, $z = 6$, find the numerical value of:

1. $4b^2 - 3c^2 + d^2$.
2. $\dfrac{4b^2 - a^2}{2cd}$.
3. $\dfrac{3x^2 - y^2}{4y + d} + 2ax - 3ay$.
4. $\left(\dfrac{b}{c}\right)^4 + \left(\dfrac{c + d}{y^2}\right)^3 - \dfrac{d - c}{y^3}$.
5. $4(x^2 - a^2) - 2(y^2 - c^2) - 3(z^2 - d^2)$.
6. $a^2(b^2 + c^2) + b^2(a^2 + c^2) + c^2(a^2 + b^2)$.
7. $4(b + c)(b - c)^2 - 3(x + y)(x - y)^2$.
8. $(b + c)^2 + b(c + d)^2$.
9. $(3a + b)(3a^2 - b^2) - (2x + y)(2x^2 - y^2)$.
10. $\sqrt{y^2 - x^2} + \sqrt[3]{z^2 - d^2} + \sqrt[4]{x^3 + c^2 + d^2 + 4b}$.
11. $\sqrt[3]{3ayz^2 + 2c^2y^2 + 4b^3x - 22xy}$.
12. $\sqrt[5]{d^4 - c^2x^2 + b^2dy}$.

II. ADDITION

A

2. Add the following:

1. $2x$, $5x$.

2. $7x$, $-3x$.

3. $-3a$, $-2a$.

4. $-3a$, $2a$.

5. $5m$, $-4m$, $-3m$.

6. 4×6, 6×6.

7. -9×5, 12×5.

8. 7×15, -5×15.

9. $4x^2 - 3x^2 - 5x^2$.

10. $\frac{3}{4}a - \frac{1}{2}a + 2a$.

11. $6bx - 7bx - 4bx + 5bx$.

12. $ab + ac + ad$.

B

Add the following:

1. $4a^2 + 3b - d + c$, $c - d - b + a^2$,
$2d - c + 2a^2 - 2b$, $d - e - c + b + 2a^2$,
$a^2 - b - c - d$.

2. $ab + c + b$, $2ab - c + 2b$, $2b - d^2 - c - ab$,
$2c - 3ab + b + d^2 + 3bc$.

3. $2b^3 - a^2b - 15a^3$, $9b^3 - 7ab^2 - 11a^2b - 4a^3$,
$6a^3 - 4ab^2 + 9b^3$, $6a^2b - 2a^3$, $12a^2b - b^3$.

4. $a^3 + 3a^2b + 3ab^2 + b^3$, $- 5ab^2 + 3a^2b - b^3 + 3a^3$,
$3ab^2 - 5a^2b + 3b^3 - 3a^3$, $- 5b^3 + 2a^2b - 4a^2$
$+ 3ab^3$, $7a^3 + 6b^3 - 5a^2b + 5ab^2$, $4a^2b + 3ab^2$
$- 3b^3 + 4a^3$.

5. $a^5 + 5a^4b + 6a^2b^2c - 7ab$,
$6a^5 - a^4b - 6a^2b^2c + 10ab$,
$- 2a^5 + 4a^4b + 12a^2b^2c - 10ab$,
$5a^5 - 16a^4b - 11a^2b^2c + 13ab$,
$- 10a^5 + 8a^4b + a^2b^2c - 6ab$.

6. $a^3 - 2b^3 - 11a^2b - 4ab^2 - bc^2$,
$4b^3 - abc + 6c^3 + 9a^2b$,
$- ac^2 + ab^2 + c^2 - 2b^3 - a^3 + 4bc^2 + 3ab^2$,
$2a^2b - 3bc^2 - 7c^3 + 3abc$.

7. $2a + \sqrt{x} + \sqrt{y} + z$, $2z - 3a - 2\sqrt{x} - 3\sqrt{y}$,
$2a - 3\sqrt{x} - \sqrt{y} + 3z - v$,
$5a - \sqrt{x} - \sqrt{y} + z + v$,
$5\sqrt{x} - 6a + 4\sqrt{y} - z + 2v$.

8. $2\sqrt{x-y} - 3x + 3\sqrt{x}$, $3\sqrt{x-y} - x - \sqrt{x}$,
$2x + \sqrt{x} - \sqrt{x-y}$, $3x - \sqrt{x-y} - 2\sqrt{x}$.

9. $6x^2y + \frac{1}{2}x^3y^2 - \frac{1}{3}x^4y^4$, $- 7x^2y - \frac{1}{4}x^3y^2 + \frac{1}{2}x^4y^4$,
$2x^2y + \frac{1}{8}x^3y^2 - \frac{1}{8}x^4y^4$,
$- 9x^2y + \frac{1}{16}x^3y^2 - \frac{3}{5}x^4y^4$.

10. $\frac{1}{2}x + \frac{1}{3}a^2b^2 - \frac{1}{4}xyz + \frac{1}{5}(a - b)$,
$\frac{1}{3}x + \frac{1}{2}a^2b^2 - \frac{1}{8}xyz - \frac{1}{10}(a - b)$,
$\frac{3}{4}x + \frac{5}{6}a^2b^2 + \frac{7}{12}xyz - \frac{1}{15}(a - b)$,
$- \frac{1}{4}x + \frac{7}{12}a^2b^2 + \frac{3}{4}xyz - \frac{3}{5}(a - b)$.

11. $3x^{\frac{1}{2}}y - 2x\sqrt{y} + 7$, $- 2x^{\frac{1}{2}}y + 2x\sqrt{y} - 3$,
$4x^{\frac{1}{2}}y - x\sqrt{y} + 2$, $3x^{\frac{1}{2}}y - 2x\sqrt{y} + 2$.

12. $amx + ady - ac + ax$, $mbx - bdyz + bx$.

13. $abx - cdx + mnx$.

14. $3x^4y^2 + x^{-m}y^{\frac{3}{2}} - 2x^3$, $-5sx^4y^2 + 2r^2x^{-m}y^{\frac{3}{2}} + 6x^3$,
$2a + x^4y^2 - b^2x^{-m}y^{\frac{3}{2}}$, $+ 3a - 2x^3 + cy$.

15. $2x^{\frac{1}{2}}y^{\frac{1}{2}} - 3x^{-1} + 2ba^{-m}b^{-m}$,
$2ax^{\frac{1}{2}}y^{\frac{1}{2}} - 2x^{-1} + 2ba^{-m}b^{-m}$,
$x^{\frac{1}{2}}y^{\frac{1}{2}} - 3ax^{-1} + bxa^{-m}b^{-m}$.

16. $(a + b - c)x + (a - b - c)y + (b + c - a)x$
$+ (b - c - a)y + (c + a - b)x + (c - a - b)y$
$+ (a + b + c)y$.

17. $ax + x^2y + c + y$, $2x^2y + 3axy - ac - ay$,
$3ax - bx^2y - bc - y$,
$2y + 3axy - x^2yz - c + 2y$.

18. $2\sqrt{x - y} + a\sqrt{x + y} + b\sqrt{x}$,
$3\sqrt{x - y} + 2a\sqrt{x + y} + c\sqrt{x}$,
$a\sqrt{x} - 3\sqrt{x - y} - 3\sqrt{x + y}$
$- 5\sqrt{x - y} - b\sqrt{x + y}$.

19. $2(a + b - c) + a\sqrt{a - m} + b\sqrt{m + n}$,
$- (a + b - c) + 3b\sqrt{a - m} + \sqrt{m + n}$,
$b\sqrt{a - m} - (a + b - c) + a\sqrt{m + n}$,
$- a\sqrt{a - m} - 2b\sqrt{m + n} - (a + b - c)$.

20. $(a + b + c)(x + a) + (b - c)(x + a)$
$+ (c + d)(x + a)$.

21. $(2a - b + c)\sqrt{x^2 - y^2} + (3a + b - c)\sqrt{x^2 - y^2}$
$+ (2a - b - c)\sqrt{x^2 - y^2}$.

22. $(\sqrt{a - b} + x)\sqrt{a + b} + (2\sqrt{a - b} + c)\sqrt{a + b}$
$+ (x - c - 3\sqrt{a - b})\sqrt{a + b}$.

III. SUBTRACTION

A

3. Subtract:

1. $5x$ / $2x$

2. $2x$ / $5x$

3. $6a$ / $-7a$

4. $-5x^2$ / $-4x^2$

5. 0 / $-6m$

6. $5a + 7$ / $4a + 3$

7. $a - 6$ / $3a - 5$

8. $3m^2 - n$ / $4m^2 - n$

9. $5ar^2 - ar$ / $2ar^2 - ar$

10. $\frac{1}{2}a$ / $\frac{3}{4}a$

11. $3.2x - .3y$ / $3x + .5y$

12. $-3a$ / $-.7a$

13. $3.14a$ / $.02a$

14. $.02a$ / $3.14a$

15. $.02a$ / $-3.14a$

16. $\frac{2}{3}h$ / $-\frac{1}{2}h$

17. $-\frac{1}{6}b$ / $\frac{5}{4}b$

B

1. From $6a^2 + 7b^2 + 7c^2 + 6bc$
take $4a^2 - 6b^2 + 8bc - 6ab + 6c^2$.

2. From $4x^3 + z^3 + x^2y + 3xz^2 + 12xyz$
take $3x^3 - 11y^3 + 3z^3 - 2x^2y - 4xz^2 + 16xyz$.

3. From $11x^4 - 2x^3 + 3x^2 - 8x$
take $3x^4 - x^2 + 7x - 14$.

4. From $4a^3b + 7a^2b^2 - 2b^4$
take $2a^4 - 7a^2b^2 - 2ab^3 + 3b^4$.

5. From $x^3 + x^2 + 64$ take $x^3 - 4x^2 + 15x + 2$.

6. From $3x^4 - 2x^3 - x^2 + x + 1$
take $x^4 - 2x^3 + 2x^2 + 2x + 1$.

7. From $12x^5 + 5x$ take the sum of
$4x^4 - 2x^3 + 3x^2 + 9$
and $6x^5 + 2x^4 + 2x^3 + 5x - 3$.

8. From $10a^4 - 4a^3x - 2a^2x^2 - ax^3$ take the sum of
$a^4 - 2a^3x + x^4$, $2a^4 + 5a^3x + 3a^2x^2 - 5x^4$,
and $6a^4 + a^3x + a^2x^2 - 4ax^3$.

9. From the sum of $12x^5 + 4xy^4 + y^5$,
$2x^5 - 4x^4y - xy^4 + 3y^5$,
and $6x^4y + 2x^2y^3 - 3xy^4$, take the sum of
$6x^5 + 2x^2y^3 - y^5$, $x^5 - 2x^4y + x^3y^2 + 2y^5$,
and $6x^5 + 4x^4y - 2x^3y^3 + 3y^5$.

10. From the sum of $3x^2y^2 - xy^3 - y^4$,
$2x^2y^2 - xy^3 + 2y^4$, $x^2y^2 + 3xy^3 + y^4$ take the sum
of $2x^2y^2 + 3xy^3 + y^4$, $-3x^2y^2 - 2xy^3 + y^4$,
$-x^2y^2 - xy^3 - y^4$.

11. From $(x + y + z)\,a + (2x + y)\,a$ take $(x - z)a$.

12. From $(3a + b - c)(x + y) + (a - b)(x + y)$
take $(a - b - c)(x + y)$.

13. From $(2a - 3x)a^2b^2 + (a + x)a^2b^2 - 3a^2b^2x$
take $(a - x)a^2b^2$.

IV. REMOVAL OF PARENTHESES

4. Remove parentheses, combining terms when possible:

A

1. $2x + (a - x)$.

2. $3a - (2a - 3)$.

3. $(2a - 3b) + (a + b)$

4. $(3x + 2y) - (3x - y)$.

5. $18 - (3 + 2)$.

6. $a - (-a - 3)$.

7. $16 - (5 + 3 \times 2)$.

8. $(-a^2 + 2a - 3) - (2a^2 + 2a - 4)$

9. $y^2 - [3y^2 + (y^2 + 3)]$.

10. $5m - [m - (3 - 2m)]$.

B

1. $2a^2 - (3a^2 + 2b + c) + a^2$
$-(2b - c) + 2b - (3a^2 + 3b + c)$.

2. $3x - (2a + x - \overline{a + b}) + 3a$
$- (2x - a - b) + 2b - [a - b - (x + a)]$.

3. $2a - (a - b - \overline{c + b}) - b + b$
$- [a - c - (2c - b)] - 2b$
$- [a + b - \overline{a + c} + b]$.

4. $x^4 - [4x^3 - \{6x^2 - (4x - 1)\}]$
$- (x^4 + 4x^3 + 6x^2 + 4x + 1)$.

5. $a - [5b - \{a - (5c - \overline{2c - b} - 4b) + 2a$
$- (a - \overline{2b + c})\}]$.

6. $2a - (3b + 2c) - [5b - (6c - 6b) + 5c$
$- \{2a - (c + 2b)\}]$.

7. $15x - \{4 - [3 - 5x - (3x - 7)]\}$.

8. $2x - (5y - \overline{3z + 7})$
$- [4 + \{x - (3y + 2z + 5)\}]$.

9. $a - [3b + \{3c - (d - b) + a\} - 2a]$.

10. $2a - \{3b + (4c - \overline{3b + 2a})\}$.

11. $2a - 3(a - \overline{b - c}) - [a - (b - \overline{a + b} + c)$
$+ d - 2a] - (b - c)$.

12. $- 10(a - 6[a - \overline{b - c}]) + 60[b - (c + a)]$.

13. $- 2[a - \{a + b - \overline{a - b} - b\}$
$- \{2a - (3b + a - \overline{4a - b})\}]$.

14. $a - 2(b - c) - [- \{ - (4a - b - c$
$- 2\{a + b + c\})\}]$.

15. $- 5\{8a - 2b - (c + a) - [- (a + b - c)$
$+ 2a - b]\}$.

16. $- 2\{ - [- (2x - y)]\} + \{ - 4(- [x - 2y])\}$.

17. $a - [a - \overline{2 - 3a} - (2b - \overline{4a + 2b})]$
$+ [3a - \{6a - (2a - 3b)\}]$.

18. $1 - [1 - \overline{1 - 4x} - \{2x - \overline{3 - 5x}\}]$
$- \{2 - (5x - 4)\}$.

19. $2x - [3y + (2y - z) - 4z$
$+ \{2x - (3y - \overline{z - 2y})\}]$.

20. $4(1 + 2x) - 2(3x + 2[2x - \overline{4 - 1}])$.

21. $84 - 7[- 11x - 4\{ - 17x + 3(8 - \overline{9 - 5x})\}]$.

Bracket like powers of x, placing a minus sign before each compound expression.

22. $b - 2ax^3 - bx^3 - ax - cx^2$
$+ bx + cx^3 - 3ax^2 + bx^2 - cx - a$.

23. $c - ax - ax^2 + cx - bx^2 - bx + cx^2 - a - b$.

24. $x^4 - 2x^3 - ax^2 - 2ax^4 - 3ax^3 - bx^2 - x - a - 1$.

25. $x^2\sqrt{a + 1} - 3ax - \sqrt{a + 1} - 2ax^2 + x\sqrt{2} + \sqrt{2}$.

26. $ax - 2x + bx^2 - 2x^2 - x^3 + cx^3$.

V. MULTIPLICATION

5. Multiply as indicated:

A

1. $a \times 2$.
2. $a \times -2$.
3. $-a \times -2$.
4. -3×-5.
5. $a \times a$.
6. $a \times 2a^2$.
7. $y^2 \times 2y$.
8. $3xy \times -2x^2y$.
9. $b^3 \times b$.
10. $b^n \times b$.
11. $-3m^2n \times 2m^3n$.
12. $(-2xy)(3x^2y)(-xy^2)$.
13. $(-5a)(-2a^n)$.
14. $(3a - b)a$.
15. $(4rs^2 + 2xy^2)(-3rx)$.
16. $(ab - ac - bc)(abc)$.
17. $(a^{n-2} + a^{n+2})a^2$.
18. $(x^{n-1} - x^{n-2})x^{2-n}$

B

1. $(2x + 5)(3x + 2)$.
2. $(2a - 3)(3a - 2)$.
3. $(4x - 3)(3x + 1)$.

4. $(2m^2 - 5)(m^2 - 4)$.
5. $x^2 + ax + a^2$ by $x - a$.
6. $2x^2 - 4bx + 3b^2$ by $2x - 3b$.
7. $4x^3 - 3x^2 - 2x + 5$ by $3x - 5$.
8. $3a^3 + 5a^2 - 7a + 3$ by $2a - 3$
9. $5a^3 - 3a^2b - 5ab^2 + 4b^3$ by $3a + 5b$.
10. $2y^3 + 3by^2 - 2b^2y - 3b^3$ by $4y - 3b$.
11. $x^4 - ax^3 + a^2x^2 - a^3x + a^4$ by $x + a$.
12. $3x^4 - 2x^3y + 4x^2y^2 - 7xy^3 + 16y^4$ by $2x - 3y$.
13. $1 + 4x - 10x^2$ by $1 - 6x + 3x^2$.
14. $x^3 - 4x^2 + 11x - 24$ by $x^2 + 4x + 5$.
15. $x^3 + 4x^2 + 5x - 24$ by $x^2 - 4x + 11$.
16. $x^3 - 7x^2 + 5x + 1$ by $2x^2 - 4x + 1$.
17. $x^3 + 6x^2 + 24x + 60$ by $x^3 - 6x^2 + 12x + 12$.
18. $x^3 - 2x^2 + 3x - 4$ by $4x^3 + 3x^2 + 2x + 1$.
19. $x^4 - 2x^3 + 3x^2 - 2x + 1$.
    by $x^4 + 2x^3 + 3x^2 + 2x + 1$.
20. $a^2 + b^2 + ab - 3a + 3b + 9$ by $a - b + 3$.
21. $4x^2 + 9y^2 + z^2 - 6xy - 2xz - 3yz$ by $2x + 3y + z$.
22. $x^2 + 4y^2 + 3z^2$ by $x^2 - 2y^2 - 3z^2$.
23. $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$.
24. $a^2 + b^2 + c^2 + ab + ac - bc$ by $a - b - c$.
25. $a^3 + 2a^2b + 4ab^2 + 8b^3$ by $a^2 - 4ab + 4b^2$.
26. $3x^4 - 2x^3y + 4x^2y^2 - 7xy^3 + 16y^4$ by $2x - 3y$.
27. $x^5 - 2x^2 + 3$ by $2x^2 - 3x - 1$.

Type Forms in Multiplication

6. The square of the sum of two quantities.

Square $(a + b)$. State a rule for this operation.

Expand by inspection:

1. $(a + 2)^2$.
2. $(a + 3)^2$.
3. $(a^2 + 1)^2$.
4. $(a + 5)^2$.
5. $(2a + 3x)^2$.
6. $(3x + 2)^2$.
7. $(2x^2 + y)^2$.
8. $(4x + 1)^2$.
9. $(3x + 2y^2)^2$.
10. $(x^{\frac{1}{2}} + 2y)^2$.
11. $(x^{-1} + 4y^{-1})^2$.
12. $(2x^{-2} + 5y^{-2})^2$.
13. $(40 + 1)^2$.
14. $\overline{23}^2$.
15. $\overline{62}^2$.

7. The square of the difference of two quantities.

Square $(a - b)$. State a rule for this operation.

Expand by inspection:

1. $(2a - 3x)^2$.
2. $(3a - 2x)^2$.
3. $(a^2 - b^2)^2$.
4. $(4a - 3x)^2$.
5. $(5x - y)^2$.
6. $(x - 2y)^2$.
7. $(5x^2 - 2x)^2$.
8. $(3x - 1)^2$.
9. $(a^{-1} - b^{-1})^2$.
10. $(a^{-2} - b)^2$.
11. $(2a^{-3} - 1)^2$.
12. $(x^{-1} - 2y^{-1})^2$.
13. $(30 - 1)^2$.
14. $\overline{39}^2$.

8. The product of the sum and the difference of two quantities.

Multiply $(x + y)$ by $(x - y)$. State a rule for securing the product.

Find by inspection the products of the following:

1. $(a - b)(a + b)$.
2. $(a + 2)(a - 2)$.
3. $(2a - 3x)(2a + 3x)$.
4. $(5x + 2)(5x - 2)$.
5. $(3x - 6)(3x + 6)$.
6. $(5x - 2y)(5x + 2y)$.
7. $(x^2 - 4)(x^2 + 4)$.
8. $(3x^2 - 4)(3x^2 + 4)$.
9. $(8a^3 - 1)(8a^3 + 1)$.
10. $(x + 2)(x - 2)(x^2 + 4)$.
11. $(x - 3)(x + 3)(x^2 + 9)$.

12. $(a + b)^2(a - b)^2$.

13. $(x^{-1} - y)(x^{-1} + y)$.

14. $(m^{-2} - 1)(m^{-2} + 1)(m^{-4} + 1)$.

15. $a^n - b^{n+1})(a^n + b^{n+1})$.

16. $[(x + y) - z][(x + y) + z]$.

17. $[(x - 2) + y][(x - 2) - y]$.

18. $[3 + (a + b)][3 - (a + b)]$.

19. $(a + b + c)(a + b - c)$.

20. $(x + y + 2)(x + y - 2)$.

21. $(4 - a - b)(4 + a + b)$.

22. $(x + y - 5)(x - y + 5)$.

23. $(x^2 + x + 1)(x^2 - x + 1)$.

24. $(x^2 - xy + y^2)(x^2 + xy + y^2)$.

25. $(20 + 1)(20 - 1)$.

26. $(91)(89)$.

27. 102×98.

9. The product of two binomials having a common term. Multiply $(x + a)$ by $(x + b)$. State a rule for products of this type.

Find by inspection the following products:

1. $(a + 4)(a + 3)$.

2. $(x + 3)(x + 2)$.

3. $(b + 5)(b + 6)$.

4. $(x + 4)(x + 3)$.

5. $(m + 12)(m + 1)$.

6. $(x + 5)(x + 1)$.

7. $(a - 3)(a - 5)$.

8. $(y - 2)(y + 3)$.

9. $(x - 4)(x + 1)$.

10. $(h - 3)(h - 5)$.

11. $(y^2 + 2)(y^2 + 1)$.

12. $(g^3 + 1)(g^3 - 2)$.

13. $(x^{-2} - 1)(x^{-2} + 1)$.

14. $(x^{\frac{1}{2}} + 6)(x^{\frac{1}{2}} - 5)$.

15. $(x^{-1} + 3)(x^{-1} + 2)$.

16. $(x^{\frac{1}{3}} + 5)(x^{\frac{1}{3}} - 1)$.

17. $(x^{-3} + 6)(x^{-3} + 2)$.

18. $(2x + y)(2x + 3y)$.

19. $(3a^2 - 5)(3a^2 - 12)$.

10. The square of any polynomial.

Square $(a + b + c)$. State a rule for the square of any polynominal.

Expand by this rule:

1. $(a + b - c)^2$.

2. $(a + b + 2)^2$.

3. $(2x - y + 1)^2$.

4. $(b - c - 1)^2$.

5. $(2 - 3x - 4x^2)^2$.

6. $(x^2 + xy - y^2)^2$.

7. $(1 - 3x + 3x^2)^2$.

8. $(x^2 + 3x + 1)^2$.

9. $(3x^2 - 2x + 1)^2$.

10. $(4x^2 + 2x + 1)^2$.

11. $(2x^3 + x^2 - x - 1)^2$.

12. $(a^3 - a^2 - a - 1)^2$.

13. $\left(a^3 - \frac{a^2}{2} - a - 1\right)^2$.

VI. DIVISION

A

11. Divide as indicated:

1. $2a)\underline{8ab}$.

2. $2c^2)\underline{12c^5}$.

3. $a)\underline{-a^2}$.

4. $-b)\underline{b^3}$.

5. $-x^2)\underline{-x^3y}$.

6. $-6a)\underline{3a^4}$.

7. $2ab)\underline{8ab^2}$.

8. $-3xyz)\underline{-9x^2y^3z^4}$.

9. $15ay^2)\underline{-45a^3y^3}$.

10. $a^n)\underline{a^m}$.

11. $x^a)\underline{x^{2a}}$.

12. $-x^{3a})\underline{x^{4a}}$.

13. $a^2b)\underline{a^{2n}b^n}$.

14. $c^xm^y)\underline{c^{3x}m^{2y}}$.

15. $a^5 \div a^{-2}$, $a^{-5} \div a^2$, $a^{-5} \div a^{-2}$.

16. $x^{\frac{3}{2}} \div x^{\frac{1}{2}}$, $x^{\frac{1}{2}} \div x^{\frac{3}{2}}$.

17. $4a^2b^2 \div 2a^{-1}b^{-1}$.

18. $a^{\frac{3}{4}}b^{\frac{2}{3}} \div a^{\frac{1}{2}}b^{\frac{1}{3}}$.

19. $a^2b^{-2}c^{\frac{2}{5}} \div a^{2m}b^{4m}c^{\frac{1}{2}}$.

20. $81a^2b^3c \div 27a^3b^3c^{-6}$.

21. $x^2 - 5x$ by x.

22. $4m^2 + 6m^3$ by $2m^2$.

23. $3a + 6a^2 - 12a^4$ by $3a$.

24. $-x^6 + x^3 - x^2$ by $-x^2$.

25. $18xy^4 - 9x^2y^5 - 27x^3y^2$ by $-9xy^2$.

B

Divide the following:

1. $x^3 + 3x^2 - 23x + 35$ by $x + 7$.

2. $x^3 - 12x^2 + 27x + 40$ by $x - 5$.

3. $2a^3 - 7a^2 - 3a + 18$ by $2a + 3$.

4. $3a^3 + 17a^2 - 43a + 20$ by $3a - 4$.

5. $3x^3 + x^2y - xy^2 + 4y^3$ by $3x + 4y$.

6. $x^4 - 8x^3y + 21x^2y^2 - 16xy^3 - 7y^4$
by $x^2 - 5xy + 7y^2$.

7. $x^4 - 9ax^3 + 12a^2x^2 + 35a^3x + 15a^4$
by $x^2 - 4ax - 3a^2$.

8. $4a^4 - 16a^3b - 4a^2b^2 + 40ab^3 + 25b^4$
by $2a^2 - 4ab - 5b^2$.

9. $4x^4 - 15ax^3 + 26a^2x^2 - 23a^3x + 8a^4$
by $4x^2 - 7ax + 8a^2$.

10. $5x^4 - 14x^3y + 31x^2y^2 - 22xy^3 + 12y^4$
by $5x^2 - 4xy + 3y^2$.

11. $2x^4 - 2x^3y - 5x^2y^2 + 4xy^3 + 5y^4$
by $2x^2 - 6xy + 5y^2$.

12. $a^5 - 5a^4b + 11a^3b^2 - 14a^2b^3 + 9ab^4 - 2b^5$
by $a^2 - 3ab + 2b^2$.

13. $a^2 + 4b^2 + 9c^2 - 12bc - 6ac + 4ab$
by $3c - 2b - a$.

14. $x^5 - 4x^4 + 3x^3 + 3x^2 - 3x + 2$ by $x^2 - x - 2$.

Divide the following:

15. $x^2 - y^2 - z^2 + 2yz$ by $y - x - z$.

16. $x^2 - 4y^2 - z^2 - 4yz$ by $2y + z - x$.

17. $a^9 - b^9$ by $a^3 - b^3$.

18. $a^2b^2 + 2abc^2 - a^2c^2 - b^2c^2$ by $ab + ac - bc$.

19. $xy^3 + 2y^3z - xy^2z + xyz^2 - x^3y - 2yz^3 + x^3z - xz^3$ by $y + z - x$.

12. Divide to five terms:

1. $2 - 3x + 4x^2$ by $1 + 2x - 5x^2$.

2. $1 - 3x^2$ by $2 - 3x - 2x^2$.

3. $a - 1$ by $1 - 2a + 2a^2$.

4. 1 by $1 - 2a + 4a^2$.

5. $1 + 3x$ by $1 - 2x$.

6. a by $1 + a$.

13. Special quotients.

Divide the following:

1. $x^2 + 5x + 6$ by $x + 2$.

2. $x^2 - x - 12$ by $x - 4$.

3. $2x^2 + 5x + 3$ by $x + 1$.

4. $x^2 - 4x + 4$ by $x - 2$.

5. $6a^2 - 5a - 21$ by $2a + 3$.

6. $a^2 - b^2$ by $a - b$.

7. $a^3 - b^3$ by $a - b$.

8. $a^2 + 4ab + 4b^2$ by $a + 2b$

9. $x^2 - 4y^2$ by $x + 2y$.

10. $x^3 + y^3$ by $x + y$.

11. $x^2 + 27$ by $x + 3$.

12. $a^4 - b^4$ by $a - b$.

13. $a^3 - b^3$ by $a - b$.

14. $a^3 + b^3$ by $a + b$.

15. $a^6 - b^6$ by $a - b$.

16. $a^5 + b^5$ by $a + b$.

17. $a^6 + b^6$ by $a^2 + b^2$.

18. $a^{12} - b^{12}$ by $a^3 - b^3$.

19. $a^{12} + b^{12}$ by $a^4 + b^4$.

20. $27a^3 - 64b^3$ by $3a - 4b$.

21. $32a^5 + 243b^5$ by $2a + 3b$.

22. $x^{10} + y^{10}$ by $x^2 + y^2$.

23. $128a^{14} + 2187b^{21}$ by $2a^2 + 3b^3$.

24. $a + b$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}}$.

25. $a^{-4} - b^{-4}$ by $a^{-1} - b^{-1}$.

26. $1 - 32x^{-5}$ by $1 - 2x^{-1}$.

Find exact divisors for:

27. $a^{15} + b^{15}$.

28. $32x^{-5} + 243y^5$.

29. $216a^3 + 8b^3$.

30. $a^{-3} + b^{-3}$.

31. $27a^6 - 8b^3$.

Review Exercises

A

14.

1. Add $9x^4 + 2x^3 + x^2 + 12x + 1$, $x^4 + 9x^3 + 7x^2 + x - 9$, $8x^4 - 7x^3 - x^2 - 15x + 10$, and $2x^4 + 12x^3 + 2x^2 + 7x + 6$.
2. From the sum of $16x^3 - 2x - 12$, $-11x^3 - 7x^2 + 1$, and $9x^3 - x^2 - x + 1$ take $2x^3 + 2x^2 - x + 2$.
3. Reduce to its simplest form $3a - [2a - 2\{a - (a - 1)\} + 2]$.
4. If $a = 2$, $b = 3$, $c = 1$, find the value of $\frac{a^2 - b^2}{c} + \frac{2ab}{\sqrt{3b}}$.
5. Multiply $x^6 - a^2x^4 + a^4x^2 - 2a^6$ by $2x - 3a$.
6. Divide $3x^3 - 2x^2y - 5xy^2 + 4y^3$ by $3x + 4y$.
7. Divide $x^6 - y^6$ by $x^2 - y^2$.
8. Find the continued product of $x + y$, $x - y$, $x^2 + xy + y^2$, and $x^2 - xy + y^2$.

B

1. If $a = 5$, $b = 3$, $c = 1$, find the value of $\sqrt{5ab^2} + \sqrt[3]{9bc} - 2\sqrt[4]{3a + b - 2c}$.

2. Square $x^3 - 2x^2 - 3x$ by inspection, and give the rule for the same.

3. Divide $a^6 - 2a^3b^3 + b^6$ by $a^2 - 2ab + b^2$.

4. Reduce to its simplest form $-2a - [3x + \{3c - (4y + 3x + 2a)\}]$.

5. Multiply $x^3 + 2x^2 + 2x + 1$ by $x^3 - 2x^2 + 2x - 1$.

6. Add $9x^2 - 7x + 5$, $-14x^2 + 15x - 6$, and $20x^2 - 40x - 17$.

7. Add $\frac{1}{2}a^3 - 2a^2b - \frac{3}{2}b^3$, $\frac{3}{2}a^2b - \frac{3}{4}ab^2 + 2b^3$, and $-\frac{3}{4}a^3 + ab^2 + \frac{1}{2}b^3$.

8. $(x^{10} - y^{10}) \div (x^2 \pm y^2)$. Use the proper sign in the divisor for exact division, and give the quotient by inspection.

C

From College Entrance Examinations

1. Simplify $3x^2 - [7x - 2 - (2x - 1)(3 - x)]$.

2. On a certain day the following hourly temperatures were recorded; find the average temperature: 7 A.M. $-8°$, 8 A.M. $-3°$, 9 A.M. $0°$, 10 A.M. $+5°$, 11 A.M. $+14°$, 12 M. $+16°$.

3. Multiply: $(3x - 2)(3x + 5)$, $(m^a + n^b)(m^a + n^b)$, and $(3y + 1)(3y - 1)$ and give the special rules of multiplication used.

4. Add $5a - (3b - 2c)$ and $-(3b - 6a) - 10a$; from the sum subtract $-4a - (3c + b)$.

5. Divide $x^5 + 50 - 70x + 37x^2$ by $10 - 2x + x^2$.

6. What is the difference in the values of $\frac{1+x}{1+y}$ and $1+x-y$ when $x = .02$ and $y = .03$? The answer is to be given correct to four places of decimals.

7. Multiply: $(x+a)\,(x-a)\,(x^2+2a^2)\,(x^4-a^2x^2-2a^4)$.

8. Simplify $2x^2-3-(3x+3x^2)-x(x^2-3)-(x+1)(2-x^2)$ and subtract the result from $5-2x$.

VII. SIMPLE EQUATIONS

15. Solve the following equations:

1. $4x + 3 = 11$.

2. $2x + 1 = x + 4$.

3. $7x - 2x + 4 = 2x - 7 + 2$.

4. $16 + 2(4y - 7) - 12y = 0$.

5. $2(x + 1) - 3 = 3(x - 1)$.

6. $(n - 4)(n + 8) = 7 - (3 - n)(n + 5)$.

7. $(2x + 5)(5x - 4) - 5x = (10x - 3)(x + 1) + 8$.

8. $x + 2 - [x - 8 - 2\{8 - 3(5 - x) - x\}] = 0$.

9. $.25x - 2 = .2x + 3$.

10. $(8a + 5)(2a + 7) - (4a - 3)(4a + 3) = 0$.

11. $.2x + 3 - .04x = 3.8$.

12. $x + 2a = 5a$.

13. $ax - 3a = 4a$.

14. $2y = 2a + 4b$.

15. $ay - ac = 3a$.

16. $ax + bx = a + b$.

17. $15(x - a) - 6(x + a) = 3(5a - 3x)$.

18. If 12 is added to 4 times a given number, the sum is 32. What is the number?

19. If 5 times a given number is diminished by 8, the result is 27. Find the number.

20. Find two numbers whose sum is 30 and whose difference is 6.

21. Find two consecutive numbers whose sum is 35.

22. Find three consecutive numbers whose sum is 69.

23. Find two consecutive even numbers whose sum is 50.

24. Divide \$154 between A and B in such a way that B shall receive \$24 more than A.

25. A rectangular field is 5 times as long as it is wide. The length of the fence around the field is 180 yards. What are the length and the width of the field.

26. The tens' digit of a number is 3 times the units' digit, and the number exceeds 7 times the sum of its digits by 9. What is the number?

27. A produce dealer sold 8 dozen cases of eggs at \$12 per case and made a profit of \$86.40. What did the dealer pay for the eggs.

28. Into what two amounts can \$2800 be divided so that the annual interest on one part at 4 per cent shall be double the annual interest on the other part at 5%?

29. Find two consecutive numbers, the difference of whose squares is 57.

30. A field is twice as long as it is wide. By increasing its length 20 rods and its width 30 rods, the area will be increased 2200 square rods. What are its dimensions?

31. The height of the big tree, *Wawona*, in California is 8 ft. more than 9 times its diameter. If the height is 260 ft., what is the diameter of the tree?

32. Two men travel toward each other from points which are 150 miles apart at rates of 5 and 15 miles an hour respectively. In how many hours will they meet?

33. A man invests \$1000, part of it at 4% and the rest at 5%, and the total income is \$48. How was the money divided?

34. Each side of a square is increased by a feet, thereby increasing its area b square feet. Find the side of the original square.

35. The sum of the two digits of a certain number is 12, and the number is 2 less than 11 times its tens' digit. What is the number?

36. A merchant has tea worth 50 cents per pound and also tea worth 65 cents per pound; how many pounds of each must he use to make a mixture of 12 lb. worth 60 cents per pound?

37. A man left $2200 to be divided among his four children, A, B, C, D. B is to get twice as much as A; C as much as A and B together, and D as much as C and B together. Find how much each received.

VIII. FACTORING

Case I

16. When all the terms have a common monomial factor. Factor the following:

1. $x^2 + xy + 2x$.

2. $2a^3 + ax^2 + ax$.

3. $3x^2y^2 - 2xy - 3xy^2$.

4. $-6cy - 18by$.

5. $2ax^2 + 4axy - 2ay^2$.

6. $6x^3y^2 - 3x^2y^3 + 3xy^4$.

7. $2x^4 + 6x^3y + 6x^2y^2 + 2xy^4$.

8. $2x^5 + 8x^4y + 12x^3y^2 + 8x^2y^3 + 2xy$.

9. $2a^2b^2 + a^3b^3 + ab^4$.

10. $15x^2 - 20xy - 5y^2$.

11. $a^{-2} + a^{-1}b^{-1} + 2a^{-1}b^{-2}$.

12. $x^{-3}y^{-1} - x^{-2}y^{-2} + x^{-1}y^{-3}$.

13. $a^{\frac{3}{2}} + ab + a^{\frac{1}{2}}b^{\frac{1}{2}}$.

14. $x^3 - x - x^2y - xy^2$.

15. $x^n + x^{n+1}$.

16. $27a^m + 54a^{2m}$.

17. $2a^{n+2} + 6^{n+3}$.

18. $16a^4b^4 + 48a^3b^2 + 16a^2b^3 - 8a^4b^2$.

19. $4a^{-5}b^{-1} + 6a^{-3}b^{-3}$.

Case II

17. Trinomial Squares: $a^2 \pm 2ab + b^2$.

A

Factor the following:

1. $x^2 + 10x + 25$.

2. $x^2 + 20x + 100$.

3. $x^{10} + 2x^5 + 1$.

4. $x^2y^2z^2 + 8xyz + 16$.

5. $x^8 + 12x^4 + 36$.
6. $a^{12} + 4a^6 + 4$.
7. $a^2 + 16a + 64$.
8. $x^{10} + 30x^5 + 225$.
9. $x^{-2} + 2x^{-1} + 1$.
10. $x^{-4} + 4x^{-2} + 4$.
11. $x + 6x^{\frac{1}{2}} + 9$.
12. $a^4x^{-2} + 8a^2x^{-1} + 16$.
13. $a^{2m} + 12a^m + 36$
14. $a^{-2}x^{-m} + 14a^{-1}x^{-\frac{m}{2}} + 49$.
15. $4x^2 + 12x + 9$.
16. $9x^2 + 42x + 49$.
17. $25x^2 + 10x + 1$.
18. $9x^2 + 48x + 64$.
19. $49x^2 + 28x + 4$.
20. $9x^2 + 6x + 1$.
21. $64x^2 + 32x + 4$.
22. $64x^4 + 16x^2y^2 + y^4$

B

Factor the following:

1. $x^2 - 28x + 196$.
2. $x^2 - 30x + 225$.
3. $x^2 - 60x + 900$.
4. $x^8 - 18x^4 + 81$
5. $x^6 - 24x^3 + 144$.
6. $x^{10} - 28x^5 + 196$.
7. $x^{-4} - 50x^{-2} + 625$.
8. $x^{-2}y^{-2} - 80x^{-1}y^{-1} + 1600$.
9. $x^{-6} - 56x^{-3} + 784$.
10. $4x^2 - 4xy + y^2$.
11. $9x^2y^2 - 12xy^2z + 4y^2z^2$.
12. $64x^4y^2 - 48x^3y^3 + 9x^2y^4$.
13. $16x^4y^4z^4 - 8x^3y^3z^3 + x^2y^2z^2$.
14. $36x^{-2} - 24x^{-1}y^{-1} + 4y^{-2}$.
15. $81x^{\frac{4}{3}} - 18x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}}$.
16. $9x^{-8} - 30x^{-4}y^{-4} + 25y^{-8}$.
17. $9x^{-2}y^{-2} - 30 + 25x^2y^2$.
18. $4x^2 - 12xy^{-1} + 9y^{-2}$.

Case III

18. The difference of two squares.

Factor the following:

1. $x^2 - y^2$.
2. $x^6 - y^2$.
3. $x^4 - y^4$.
4. $16x^2 - 25y^2$.
5. $81y^4 - 64z^4$.
6. $\frac{9}{4}a^2 - \frac{1}{9}b^2$.
7. $25b^{2n} - a^2$.
8. $25m^2 - 4n^2$.
9. $3r^3 - 75rs^6$.
10. $.01x^2 - .04y^2$.
11. $(a + b)^2 - (a - b)^2$.
12. $a^2 - 2ab + b^2 - c^2$.
13. $a^4 - 2a^2b^2 + b^4 - 1$.
14. $1 - a^2 - 2ab - b^2$.
15. $(2x + 3y)^2 - (2x - 3y)^2$.
16. $x^2 - (x + y)^2$.
17. $(3x - 2)^2 - (2x - 3)^2$.
18. $2xy - x^2 - y^2 + 1$.
19. $x^2 - 2yz - y^2 - z^2$.
20. $a^2 - x^2 + 2x - 1$.
21. $9x^2 + 6xy - 4z^2 + y^2$.
22. $a^{16} - b^{16}$ into five factors.
23. $16x^{-4} - 16y^{-4}$.
24. $x^{-\frac{2}{3}} - y^{-\frac{2}{3}}$.
25. $a^{-2} - 2a^{-1}b^{-1} + b^{-2} - c^{-2}$.
26. $9a^2 - 12ab + 4b^2 - 16x^2 - 8xy - y^2$.
27. $4x^2y^2 - (x^2 + y^2 - z^2)^2$.
28. $4(ad + bc)^2 - (a^2 - b^2 - c^2 + d^2)^2$.
29. $(a^2 + 9b^2 - c^2)^2 - 36a^2b^2$.
30. $64x^2y^2 - (x^2 + 16y^2 - z^2)^2$.

B

Trinomials in the form of $a^{2n} \pm a^nb^n + b^{2n}$, to be reduced to the difference of two squares, and factored by Case III.

Factor the following:

1. $x^4 - 6x^2y^2 + y^4$.
2. $x^4 + x^2 + 1$.
3. $a^4 - 7a^2b^2 + b^4$.
4. $16x^4 - 17x^2y^2 + y^4$.
5. $9a^4 + 21a^2b^2 + 25b^4$.
6. $49x^4 - 15x^2y^2 + 121y^4$.

7. $25x^4 + 66x^2y^2 + 49y^4$.
8. $121x^8 - 180x^4y^4 + 64y^8$.
9. $36a^4 - 28a^2b^2 + 4b^4$.
10. $64a^4 + 60a^2b^2 + 16b^4$.
11. $16x^{-4} + 36x^{-2}y^{-2} + 25y^{-4}$.
12. $a^8 - 14a^4y^4 + y^8$.
13. $n^4 + 4m^4$.
14. $h^4 + 4$.
15. $1 + 64y^4$

Case IV

A

19. Trinomials of the form $x^2 + bx + c$.

Factor the following:

1. $x^2 + 12x + 35$.
2. $x^2 + 11x + 24$.
3. $a^2 + 15a + 56$.
4. $m^2 + 16m + 63$.
5. $x^2 + 30x + 225$.
6. $x^4 + 8x^2 + 7$.
7. $x^{10} + 20x^5 + 19$.
8. $x^{-2} + 14x^{-1} + 45$.
9. $a^{-4} + 15a^{-2} + 36$.
10. $a^{-1} + 5a^{-\frac{1}{2}} + 6$.
11. $x^2 + (a + b)x + ab$.
12. $(x - y)^2 + 5(x - y) + 6$.
13. $a^4x^4 + 18a^2x^2 + 17$.
14. $x^2 + (a + 2b + c)x + (a + b)(b + c)$.

B

Factor the following:

15. $x^2 - 13x + 40$.
16. $x^2 - 7x + 6$.
17. $x^2 - 9x + 20$.
18. $x^8 - 19x^4 + 88$.
19. $x^6 - 10x^3 + 21$.
20. $x^{-2} - 24x^{-1} + 143$.
21. $a^{-\frac{2}{3}} - 12a^{-\frac{1}{3}} + 20$.
22. $a^4x^{-2} - 3a^2x^{-1} + 2$.
23. $x^4 - 21x^2 + 90$.
24. $x^{-8} - 33x^{-4} + 260$.
25. $x^{10} - \frac{7}{6}x^5 + \frac{1}{3}$.
26. $x - 15x^{\frac{1}{2}} + 50$.

C

Factor the following:

27. $x^2 + 3x - 40$.
28. $a^2 + 8a - 20$.
29. $x^4 + 3x^2 - 10$.
30. $x^2 + 3x - 54$.
31. $x^2 + 3x - 180$.
32. $x^2 + 8x - 65$.
33. $x^2 + x - 2$.
34. $x^2 + \frac{1}{6}x - \frac{1}{6}$.
35. $x^6 + x^3 - 42$.
36. $x^4y^2 + 6x^2y - 40$.
37. $x^{-4}y^{-2} + 3x^{-2}y^{-1} - 28$.
38. $x^{-\frac{1}{2}} + 2x^{-\frac{1}{4}} - 8$.
39. $x^2 - 3x - 108$.
40. $x^2 - 2x - 323$.
41. $x^4 - 7x^2 - 8$.
42. $x^{-2} - 2x^{-1} - 99$.
43. $a^4 - 4a^2 - 165$.
44. $a^2b^2 - 3ab - 40$.
45. $a^4b^2 - 2a^2b - 63$.
46. $a^{-4} - 13a^{-2} - 14$.

Case V

20. Trinomials of the form $ax^2 + bx + c$.

Factor the following:

1. $2x^2 - 3x - 2$.
2. $2x^2 + 5x + 2$.
3. $3x^2 + 2x - 1$
4. $4x^2 - 11x + 6$.
5. $5x^2 - 11x + 2$.
6. $56x^2 + 9x - 2$.
7. $56x^2 + 26x - 1$.
8. $16x^2 + 2x - 3$.
9. $56x^2 - 9x - 2$.
10. $35x^2 + 17x + 2$.
11. $6x^4 - 13x^2 + 6$.
12. $8x^6 + 2x^3 - 1$.
13. $15x^2 + 24x - 12$.
14. $55x^4 - x^2 - 2$.
15. $12x^2 - 25x + 12$.
16. $24x^2 + 18x + 3$.
17. $6x^2 + 7xy + 2y^2$.
18. $21x^2 + xy - 2y^2$.
19. $4x^4 - 2x^2y^2 - 6y^4$.
20. $6x^2y^2 - 5xyz - 6z^2$.

Case VI

21. The sum or difference of like powers.

Factor the following:

1. $x^3 + y^3$.
2. $x^3 - y^3$.
3. $x^5 + y^5$.
4. $x^5 - y^5$.
5. $8x^3 + y^3$.
6. $x^3 - 343y^6$.
7. $32x^5 + 243y^5$.
8. $128x^7 + y^7$.
9. $a^6 + 64b^6$.
10. $125a^3 + 27b^3$.
11. $27x^6 - 8y^6$.
12. $a^{-9} + b^{-9}$.
13. $8a^{-6} + 27b^{-6}$.
14. $1728a^{-3} + 64b^{-3}$.
15. $243x^{10} + y^{15}$.
16. $x^3y^3 + 8$.
17. $a^{12} - b^6$.
18. $(a + b)^3 + 1$.
19. $32x^5 - 1$.
20. $(a + 1)^3 - 8(a + 2)^3$.

Case VII

22. Expressions which can be arranged in groups of two or more terms so that all the groups shall contain a common factor.

Factor the following:

1. $a^2 - ax + ab - bx$.
2. $ax + ay - bx - by$.
3. $x^2 - 2xy + xz - 2yz$.
4. $2x^2 - 2xy - x + y$.
5. $6x^2 - 9x - 10xy + 15y$.
6. $2ax - 3bx + 2a - 3b$.
7. $xy + x + y + 1$.
8. $x^2y + 2xy - 2x^2 - 4x$.
9. $a^4 + a^2y^2 - a^2b^2 - b^2y^2$.
10. $8x^2 - 12ax - 6xy + 9ay$.
11. $x^4 - 2x^3 + 7x - 14$.
12. $x^3 + 3x^2 - 5x - 15$.
13. $x^5 + 5x^4 + x^2 + 5x$.
14. $2x^3 + x^2 + 6x + 3$.
15. $14x^3 - 6x^2 - 21x + 9$.

16. $2a^2x^2 + 3a^2x + 2b^2x^2 + 3b^2x + a^2 + b^2$.
17. $3ax - 3ab - 2x^2 + 2bx - b + x$.
18. $x^2(a - b) + x(b - a) + 2(a - b)$.
19. $2x - ax - ay - az + 2y + bx + 2z + by + bz$.
20. $x^3 + (a + 1)x^2 + (a + 1)x + a$.

Review Exercises in Factoring

23. Factor the following:

1. $a^2 - b^2$.
2. $36a^2 - 49b^2$.
3. $a^2 + 11a + 30$.
4. $a^2 - a - 6$.
5. $a^3 - b^3$.
6. $a^3 + b^3$.
7. $a^4 - b^4$.
8. $a^5 - b^5$.
9. $(a + b)^2 - (a - b)^2$.
10. $a^2 + 16a + 64$.
11. $x^2 - 8x + 16$.
12. $x^3 + 3x^2 + 3x + 1$.
13. $49x^2 + 28xy + 4y^2$.
14. $4x^2 - 12xy + 9y^2$.
15. $x^2 - a^2 - 2ab - b^2$.
16. $a^2 - 2ab + b^2 - x^2$.
17. $a^2 - 2ab + b^2 - x^2 + 2xy - y^2$.
18. $x^2 + y^2 - a^2 - b^2 - 2xy - 2ab$.
19. $(x + y)^2 - (a - b)^2$.
20. $2abx + 3aby + 2a^2b^2x + 3a^2b^2y$.
21. $a^3 + 3a^2b + 3ab^2 + b^3$.
22. $x^3 - 27$.
23. $a^2 + ac + 3ab + 3bc$.
24. $2ax - 6x - 2ay + 6y$.
25. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.
26. $x^3 + 27$.
27. $x^2 + x - 6$.
28. $x^2 - 5x + 6$.
29. $4x^2 + 12xy + 9y^2$.
30. $4x^2 + 4xy + y^2$.
31. $a^2 - b^2 + a - b$.
32. $a^3 - 8b^3$.
33. $8a^3 - b^3$.
34. $a^4 - b^8$.
35. $ax - ay + bx - by$.
36. $(a + b)^2 + 2(a + b) + 1$.
37. $a^5 - ab^4$.
38. $ax + ay + bx + by$.
39. $12a^2b - 75b^5$.

40. $x^2 + 2x + x + 2$.

41. $ax + bx + ac + bc$.

42. $x^2 - y^2 + (x - y)^2$.

43. $x^4 - y^2$.

44. $x^4 - 6x^2y^2 + y^4$.

45. $x^4 + x^2y^2 + y^4$.

46. $6x^3 - 2x^2 - 21x + 7$.

47. $8x^3 - 6x^2 - 28x + 21$.

48. $(x^2 - 2x - 1)^2 - (x^2 - 1)^2$.

49. $x^3 - 2x^2 + 3x - 6$.

50. $(a^2 + b^2 - c^2)^2 - 4a^2b^2$.

51. $a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$.

52. $(2x + y - z)^2 - (x - y + z)^2$.

53. $x^2 - 2x + 1 - y^2 - 4yz - 4z^2$.

54. $x^4y + x^2y^3 - x^3y^2 - xy^4$.

55. $(x + y)^3 - (x - y)^3$.

56. $x^2 - y^2 + (x - y)^2$.

57. $(a + 3b)^4 - (a - 3b)^4$.

58. $x^4 - 167x^2y^2 + y^4$.

59. $x^8 - 7x^4y^4 + y^8$.

60. $ax^4 - ax^2y^2 + ax^3y - axy^3$

61. $(x + y)^2(x - y) - (x - y)^2(x + y)$.

62. $x^2 - 4y^2 + 12yz - 9z^2$.

63. $x^4 + 2x^2z - 4y^2 + z^2$.

64. $8x^3 - 12x^2 + 6x - 1$.

65. $6x^2 - 7xy + 2y^2$.

66. $20x^3 + 26x^2y - 6xy^2$.

67. $64x^6 - 729y^6$.

68. $x^2 + 4ax + 4a^2 - b^2$.

69. $4a^3 - a^2 - 8a + 2$.

70. $x^4 - 27x^2y^2 + y^4$.

71. $32x^5 - 1$.

72. $\dfrac{x^2}{a^2} + \dfrac{a^2}{x^2} - 2$.

73. $x^2 + 4y^2 - 9z^2 - 4xy$.

74. $16^x + 2 \cdot 2^{2x} + 1$. **75.** $2^{5x} - 32$.

76. $acx^2 + (ad + bc)x + bd$.

77. $1 - x - x^2 + x^3$.

24. *From College Entrance Examinations*

Factor:

1. $6x^2 - 5x - 4$; $30m + 4p^2 - 25 - 9m^2$;
$x(x + 2)(2x - 1) - (x + 2)$.

2. $4x^2 - 25y^2 + 2x + 5y$; $6a^2b^2 - ab^3 - 12b^4$;
$x - 27x^4$.

3. $6x^2 - x - 77$; $\frac{x^2}{y^2} + \frac{y^2}{x^2} + 2$;
$a^3 + b^3 + a + b$.

4. $3 - 192x^6$; $a^2 - 6a - 4b^2 - 12b$;
$6x^2 + x - 15$.

5. $1 - y - z + yz$; $10 - x - 3x^2$.

6. $a^2 + 4y^2 - 9x^2 - 4ay$.

7. $4x^4 + 7x^2 - 36$; $(x + 1)(x + 2)(x + 3) - 12(x + 1)$

8. $12a^5b - 58a^3b + 40ab$;
$2(x^3 - 1) + 7(x^2 - 1)$.

9. $u^5 + v^5$; $x^4 - 4x^2 - x + 2$.

10. $6x^2 + 37x - 60$; $x^4 + x^3y - xy^3 - y^4$;
$x(x + 1)(4x - 5) - 6(x + 1)$.

11. $a^{6x} - b^{2x}$.

12. $6x^2 - 13x + 5$; $x^{12} - 64a^6$; $x^2 - a^2 + 2ay - y^2$.

13. $8x^2 - 34x + 33$; $8x^3 - 27$; $a^3 + b^3 - ab(a + b)$.

14. $a^2 - c^2 + 2ab + b^2$; $y^8 - z^8$; $m^{a+b} + m^an^a + m^bn^b + n^{a+b}$.

15. $4a^2b^2 - (a^2 + b^2 - c^2)^2$; $x^{16} + a^6xy^9$;
$a^4 - a^2b^2 + b^2 - 1$.

IX. HIGHEST COMMON FACTOR

A

25. Find the H. C. F. of:

1. $3x^2y$, $6x^2y^2$, $9xy^2$.

2. $24a^2x^3$, $56a^3x^2$.

3. $27a^6$, $45a^4b^3$, $72a^5b^2$.

4. $b^2 - 3b$, $b^2 - 9$.

5. $3ax^3 + 4bx^4$, $ax^5 - 12bx^6$.

6. $a^2 + b^2$, $a^4 - b^4$, $a^6 + b^6$.

7. $y^2 + y$, $y^2 - 1$, $y^2 - y - 2$.

8. $2x^3 - 2x$, $3x^4 - 3x$, $4x(x - 1)^2$.

9. $a^2 + 2ab + b^2$, $a^2 - b^2$, $a^3 + b^3$, $a^4 - b^4$.

10. $12(x - 1)^2$, $6(x - 1)^3$, $4(x^2 + x - 2)$, $2(x^2 - 1)$.

11. $a^2 + 4a + 4$, $a^2 + 3a + 2$, $a^2 + a - 2$.

12. $x^3 + y^3$, $x^2 - xy + y^2$, $x^6 - y^6$.

13. $a^2 + 3a - 4$, $a^2 - 7a + 6$, $a^2 - 2a + 1$.

14. $15(x + y)^2$, $12(x^2 - y^2)^2$, $9(x^2 - y^2)$.

15. $4 - a^2$, $a^2 - a - 2$, $(2 - a)^2$.

B

Find the H. C. F. of the following:

1. $a^2 - b^2$, $a^3 - b^3 - 3ab(a - b)$.

2. $a^2 + ab - 2b^2$, $a^2 + 6ab + 8b^2$.

3. $a^2 + 3ab + 2b^2$, $a^2 - ab - 2b^2$, $a^5 + b^5$.

4. $ab - ac + 2b - 2c$, $a^3 + 6a^2 + 12a + 8$.

5. $a^2 - 2a + ab - 2b$, $3a^2 - 5a - 2$.

6. $6x^2 - 5x - 6$, $2ax - 2bx - 3a + 3b$.
7. $a^2 - ab - 6b^2$, $a^2 - 2ab - 8b^2$, $a^2 - 3ab - 10b^2$.
8. $2a^2 - 13a - 7$, $2a^3 - a^2 + 3a + 2$.
9. $8x^4 - 6x^3 - 28x^2 + 21x$, $6x^5 - 25x^3 + 14x$.
10. $27x^3 + 6x - 3$, $27x^2 - 3$, $27x^2 - 18x + 3$.
11. $x^3 + 4x^2 - 47x - 210$, $x^2 - x - 42$.
12. $a^2 + ac + ab + bc$, $2a^2 - ab - 3b^2$.
13. $2x^2 - 15x + 25$, $x^3 - 5x^2 - x + 5$.
14. $2x^4 - 6x^3 + 3x^2 - 3x + 1$, $a^2x^2 - 3a^2x + a^2$.
15. $a^2x - b^2x$, $a^2 - 3ab + 2b^2$, $a^3 - ab + b^2 - a^2b$.

Lowest Common Multiple

A

26. Find the L. C. M. of:

1. $3a^2b$, $6ab^2$, $8bc$.
2. $3a^2b$, $4ac^2$, $9b^2c$.
3. $12x^2y$, $16xy^2$, $24x^2y^2$.
4. $2a^2 + 2a$, $a^2 - 1$.
5. $3a + 6$, $6a^2 - 24$, $2a - 4$.
6. $5m^2$, $3m^2 - 6m$.
7. $x - y$, $x^2 + xy + y^2$, $x^3 - y^3$.
8. $x + y$, $x^2 - xy + y^2$, $x^3 + y^3$.
9. $y - 3$, $y^2 + 3y + 9$, $y^3 - 27$.
10. $9x^2 - 12x + 4$, $9x^2 - 4$, $3x^2 + 2x$.
11. $4a^2 - 12ax + 9x^2$, $4a^2 - 9x^2$, $2a - 3x$.
12. $12a^3b^2c^2$, $16a^4bcd^2$, $15a^2b^2c^2$, $6abcd$.
13. $20a^5bcx^2y^2$, $12a^3b^3c^3d^3$, $15abc^2dxy$.
14. $b^2(b - a)^2$, $b(a^2 - b^2)$.
15. $1 - x + x^2 - x^3$, $1 + x + x^2 + x^3$, $2x - 2x^3$.

B

Find the L. C. M. of the following:

1. $x^2 - 2x + 1$, $x^2 - 3x + 2$, $x^2 + x - 2$.

2. $2(x + 1)^2$, $4(x^3 - 1)$, $3(x^2 - 1)$, $2(x^2 + x - 2)$.

3. $2x^2 - x - 1$, $2x^2 + 5x + 2$, $2x^2 + 7x + 3$.

4. $x^2 + 4x - 21$, $x^2 - 3x - 70$, $x^3 - 39x + 70$.

5. $x^3 - y^3$, $x^3 + y^3$, $x^6 - y^6$, $x^{12} - y^{12}$.

6. $x^2 - 4$, $x^2 - 7x + 10$, $x^3 - 5x^2 + 4x - 20$.

7. $x^2 + 2x - 3$, $3x^2 - 5x + 2$, $4x^3 - 4x^2 - x + 1$.

8. $x^3 + 2x^2 - 9x - 18$, $x^2 + x - 2$, $x^2 - 4x + 3$.

9. $4x^3 + 8x^2 - x - 2$, $2x^2 + 15x - 8$, $x^2 + 10x + 16$.

10. $12x^2 + 3x - 42$, $12x^3 + 30x^2 + 12x$, $32x^2 - 40x - 28$.

11. $2x^2 - x - 10$, $2x^2 + x - 15$, $4x^2 - 20x + 25$.

12. $2ax^2 - 2ax - 12a$, $3x^2 + 3x - 6$, $5x^2 - 20x + 15$.

13. $9ax^3 - 3ax^2 - 42ax$, $6x^2 - 26x + 28$, $x^2 - 4$.

14. $16x^2 - 76xy + 70y^2$, $12x^2 - 3xy - 15y^2$, $8x^2 - 20xy - 28y^2$.

15. $24ax^2 - 46axy + 20ay^2$, $12bx^2 - 27bxy + 15by^2$, $6cx^2 - 10cxy + 4cy^2$.

From College Entrance Examinations

27. Find the H. C. F. and L. C. M. of:

1. $x^3 - 1$; $2x^2 - 5x + 3$.

2. $x^2 - 5x + 6$; $x^2 - 4$; $x^3 - 3x - 2$.

3. $x^4 - ax^3 - 2a^2x^2$; $2x^3 - 2a^2x$; $3x^3 + 12ax^2 + 3a^2x$.

4. $9(1 - x)^3(2 + x)$, $3(x + 1)(x - 1)(x - 2)$, $6(x + 1)^2(x - 1)^2(2x + 3)$.

5. $x^3 - x$; $x^3 + 9x^2 - 10x$; $x^6 - x$.

6. $(xy - y^2)^3$; $y^3 - x^2y$.

7. $p^2qr(p + q)^2$; $q^2rp(p^2 - q^2)$; $r^2pq(p - q)^2$

8. $x^5 - xy^2$; $x^3 + x^2y - xy - y^2$.

X. FRACTIONS

28. Reduction to lowest terms.

A

Reduce the following fractions to their lowest terms:

1. $\frac{a}{ab}$. **2.** $\frac{a^2}{ab}$. **3.** $\frac{6x^3y}{8xy^2}$. **4.** $\frac{12x^2}{3x^3}$.

5. $\frac{-3ab}{9a}$. **6.** $\frac{-xyz}{-xy}$. **7.** $\frac{84m^3n^4}{-96m^5n^3}$.

8. $\frac{4x^2r}{8x^2 - 12x^3r}$. **9.** $\frac{x^2 - 1}{(x + 1)^2}$.

10. $\frac{m^2 - 6m}{m^3 - 7m^2 + 6m}$. **11.** $\frac{(-2abc)^3}{8ab^2c^3}$.

12. $\frac{a^2 + 4a + 3}{a^2 + 2a - 3}$.

13. $\frac{a - ab + c - cb}{1 - 3b + 3b^2 - b^3}$.

14. $\frac{x^{2a} + 2x^a + 1}{x^{2a} - 1}$.

15. $\frac{6x^2 + x - 2}{6x^2 - x - 1}$.

B

Reduce the following fractions to their lowest terms:

1. $\frac{a^2 - b^2}{b^3 - a^3}$. **3.** $\frac{(x - y)^2}{y^2 - x^2}$.

2. $\frac{n^2 - m^2}{(m - n)^2}$. **4.** $\frac{9 - a^2}{a^2 - 7a + 12}$.

5. $\dfrac{a+b-c}{c^2-(a+b)^2}$.

6. $\dfrac{4-(a+b)^2}{(a-2)^2-b^2}$.

7. $\dfrac{6x^2+7x-3}{18x^2-12x+2}$.

8. $\dfrac{x^2-(a+b)^2}{a^2-(x+b)^2}$.

9. $\dfrac{(x+y)^2-(x-y)^2}{(x-y)^2-(x+y)^2}$.

29. Reduction to integral or mixed forms.

Reduce to integral or mixed expressions:

1. $\dfrac{19}{3}$. 2. $\dfrac{16a^5}{2a}$. 3. $\dfrac{xy+a}{x}$.

4. $\dfrac{a-b}{a}$. 5. $\dfrac{4a^3+4a-6}{2a}$.

6. $\dfrac{6x^2-19x+15}{2x-3}$.

7. $\dfrac{3x}{x-3}$. 8. $\dfrac{9a^2}{3a^2-2x}$.

9. $\dfrac{15x^2-11x+6}{5x-2}$.

10. $\dfrac{2x^3+3x+2}{2x+1}$.

30. Reduction of mixed to fractional forms.

Reduce to fractional forms:

1. $\dfrac{a+b}{a}+2$. 4. $2x^2-\dfrac{2x(1+xy)}{x+y}$.

2. $a+b-\dfrac{a+b}{2}$. 5. $\dfrac{(x+y)^2}{xy}-4$.

3. $2a-b+\dfrac{a^2-b^2}{2a+b}$. 6. $2a-7-\dfrac{2a^2-a-20}{a+3}$.

7. $\dfrac{3x^2 - 6y}{x + 3} - 3x + 2y.$

8. $2x + y + 1 - \dfrac{2x^2 + 3xy + y^2 + x + y}{x + y}.$

9. $\dfrac{6a^2 + 5a + 1}{2a + 1} - 3a - 1.$ 10. $x + 3 + \dfrac{x^2 + 9}{x - 3}.$

11. $x + 2 - \dfrac{x^2 + x - 6}{x - 3}.$

12. $x + 2a - \dfrac{x^2 - x + 2a^2}{x + a}.$

31. Addition and subtraction of fractions.

Combine the following:

1. $\dfrac{x}{a} + 2a.$ 2. $\dfrac{a}{2b} - \dfrac{b}{3a}.$ 3. $\dfrac{5a}{x^2} - \dfrac{2b}{xy}.$

4. $\dfrac{2}{3a} - \dfrac{1}{2b} - \dfrac{2a + 3}{6a^2} + \dfrac{1}{2a^2} + \dfrac{3a - 2b}{6ab}.$

5. $\dfrac{1}{a - b} + \dfrac{1}{b - a}.$ 6. $\dfrac{a}{a - b} + \dfrac{b}{a + b}.$

7. $\dfrac{a}{a - x} + \dfrac{3a}{a + x} - \dfrac{2ax}{a^2 - x^2}.$

8. $\dfrac{2xy}{x^2 - y^2} + \dfrac{3y}{2x} + \dfrac{3x}{2y} - \dfrac{3x^2 - 3y^2}{2xy}.$

9. $\dfrac{1}{x^2 - 3x + 2} + \dfrac{2}{x^2 - 4x + 3} + \dfrac{4}{x^2 - 5x + 6}.$

10. $\dfrac{x - 3}{2x^2 - 5x + 2} + \dfrac{x + 2}{2x^2 - 7x + 3} - \dfrac{2x + 1}{x^2 - 5x + 6}.$

11. $\dfrac{x}{3x^2 - 7x + 2} - \dfrac{2}{3x^2 - 5x - 2} + \dfrac{3}{9x^2 - 1}.$

12. $\dfrac{2a}{a^2 - b^2} - \dfrac{a^2}{a^3 + b^3} - \dfrac{1}{a - b}.$

13. $\dfrac{1}{6x + 6} + \dfrac{1}{6 - 6x} - \dfrac{1}{3x^2 - 3}.$

14. $1 - \dfrac{x-y}{x+y} + \dfrac{2y^2}{x^2-y^2} + \dfrac{2xy}{x^2+y^2}$.

15. $\dfrac{a-b}{2x-4-xy+2y} + \dfrac{x+2}{2a-ay+2b-by}$.

16. $\dfrac{1}{x-1} + \dfrac{x+1}{x^2-3x+2} + \dfrac{x-2}{2x^2-5x+2}$.

17. $\dfrac{x-3}{x^2-2x-3} + \dfrac{x-2}{x^2-3x+2} + \dfrac{1}{x^2-1}$.

18. $\dfrac{1}{x-2} + \dfrac{2}{x^2-4x+4} + \dfrac{x^2+4}{x^3-6x^2+12x-8}$.

19. $\dfrac{1}{x^2-(x+y)^2} + \dfrac{1}{x^2-(x-y)^2} + \dfrac{1}{4x^2-y^2}$.

20. $\dfrac{1}{3x^3-27x} + \dfrac{-1}{x^2+x-12} + \dfrac{-2}{(x+3)^2}$.

21. $\dfrac{x+3}{x^2-9} + \dfrac{x-3}{x^3-27} + \dfrac{2}{x-3} + \dfrac{5}{x^2+3x+9}$

22. $\dfrac{a+b}{a-b} + \dfrac{a-b}{a+b} + \dfrac{a^2+b^2}{a^2-b^2}$.

23. $\dfrac{3}{x+a} - \dfrac{1}{x+3a} + \dfrac{3}{a-x} + \dfrac{1}{x-3a}$.

24. $\dfrac{x+2}{(x-2)(x-3)} - \dfrac{x+2}{(3-x)(x-4)} + \dfrac{1}{(4-x)(2-x)}$.

25. $\dfrac{5a+2}{(a-1)(3-a)} - \dfrac{2a+3}{(2-a)(a-1)} + \dfrac{3a-2}{(a-3)(a-2)}$.

26. $\dfrac{2}{x-3a} + \dfrac{2a}{(x-2a)^2} - \dfrac{x-a}{(3a-x)(2a-x)}$.

27. $\dfrac{x-6}{2(x^2-9x+18)} + \dfrac{x-5}{(x-5)(2-x)} + \dfrac{1}{2(x-1)}$.

28. $\dfrac{x(16-x)}{x^2-4} + \dfrac{2x+3}{2-x} - \dfrac{2-3x}{x+2}$.

29. $\dfrac{5}{3-6x} - \dfrac{5}{3+6x} - \dfrac{x}{2+8x^2}$.

30. $\frac{1}{4(1+x)} + \frac{1}{4(1-x)} + \frac{1}{2(1+x^2)}$.

31. $\frac{3-2b}{2-2x-a+ax} + \frac{2+a}{3-3x-b+bx}$.

32. $\left(2a - \frac{a+3}{2}\right) + \left(a - \frac{a-4}{3}\right) - 2a$.

33. $\frac{1}{x+1} - \left[\frac{1}{x-1} - \frac{1}{x^2-1}\right]$.

34. $\frac{2}{x+5} - \left[3 - \frac{1}{x-5} - \left(\frac{1}{x^2-25} + \frac{x}{x-5}\right)\right]$.

35. $\frac{1}{1+x} - \left[\frac{6}{1-x} - \left(\frac{2}{1+2x} - \frac{16}{2x-1}\right)\right]$.

36. $\frac{a}{(a-b)(a-c)} + \frac{b}{(b-c)(b-a)} + \frac{c}{(c-a)(c-b)}$.

37. $\frac{a+b}{(c-a)(c-b)} + \frac{b+c}{(a-b)(a-c)} - \frac{c+a}{(b-c)(b-a)}$.

32. Multiplication of fractions.

Multiply the following:

1. $\frac{12x^2}{5y^2} \times \frac{10xy}{9z^2}$.

2. $\frac{3a^3c^2d}{4xyz} \times \frac{16xz}{27a^4c^3}$.

3. $\frac{a^2b}{x^2y} \times \frac{b^2c}{y^3z} \times \frac{x^2y^2z}{abc}$.

4. $\left(-\frac{c^3}{d^3}\right)\left(-\frac{d}{c}\right)$.

5. $\left(\frac{-a}{2}\right)\left(\frac{-4}{a^2}\right)$.

6. $\frac{x+1}{y} \times \frac{4y^2}{x^2-1}$.

7. $\frac{3x^2-x}{a} \times \frac{2a}{4x-2x^2}$.

8. $\dfrac{(y-x)^2}{x+y} \times \dfrac{y}{x(x-y)}$.

9. $\dfrac{x^2-144}{x^2-9} \cdot \dfrac{x+3}{x+12}$.

10. $\dfrac{mn}{m+n}\left(\dfrac{m}{n}-\dfrac{n}{m}\right)$.

11. $\dfrac{a^4-x^4}{a^3-x^3} \times \dfrac{a^2-x^2}{a^3+x^3} \times \dfrac{a^6-x^6}{a^2+x^2}$.

12. $\dfrac{x^2+4x-5}{x^2-2x-3} \times \dfrac{x^2-x-6}{x^2-4x-5} \times \dfrac{x^2+2x+1}{x^2+x-2}$.

13. $\left(x-\dfrac{y^2}{x}\right)\left(\dfrac{x}{y}+\dfrac{y}{x}\right) \times \dfrac{xy}{x^4-y^4}$.

14. $\dfrac{a^2-x^2}{a+y} \times \dfrac{a^2-y^2}{ax+x^2} \times \left(a+\dfrac{ax}{a-x}\right)$.

15. $\dfrac{x^2-(x-y)^2}{x^2-(x+y)^2} \times \dfrac{4x^2+4xy+y^2}{6x^2+xy-2y^2} \times \dfrac{3x^2-xy-2y^2}{2x+y}$.

16. $\dfrac{3a^2-a-2}{2a^2-a-1} \times \dfrac{9a^2+12a+4}{a^2+2a-3} \times \dfrac{2a^2+7a+3}{3a^2+8a+4}$.

17. $\dfrac{9x^2-6x}{4x^2-8x+3} \times \dfrac{2x^2+3x-9}{6x^2-7x+2} \times \dfrac{2x^2+13x-7}{2x^2+6x}$.

18. $\left(4x^2+14x+\dfrac{98x-27}{2x-7}\right)\left(\dfrac{1}{6}-\dfrac{3x+29}{12x^2+18x+27}\right)$.

19. $\dfrac{a^2-b^2}{a^3-b^3} \times \dfrac{(a-b)^2}{(a+b)^2} \times \dfrac{a^4-b^4}{a^8-b^8} \times \dfrac{a^2+ab+b^2}{(a^2-b^2)^2}$.

20. $\dfrac{x^2+5xy+6y^2}{x^2-y^2} \times \dfrac{ax+bx+ay+by}{ax+3by+bx+3ay} \times \dfrac{2(x-y)}{5a(x+2y)}$.

21. $\dfrac{x^2-y^2}{x^2-3xy+2y^2} \times \dfrac{xy-2y^2}{x^2+xy} \times \dfrac{x^2-xy}{(x-y)^2}$.

22. $\dfrac{x^2-2ax+a^2}{x^2+4ax-5a^2} \times \dfrac{x^2-9a^2}{ax+2a^2} \times \dfrac{x^2+5ax}{x^2-4ax+3a^2}$.

23. $\left(b + \frac{ab}{b-a}\right)\left(b - \frac{ab}{a+b}\right)\left(\frac{b^2-a^2}{b^2+a^2}\right)$.

24. $\frac{a^5-x^5}{a-x} \times \frac{a+x}{(a-x)^2} \times \frac{a^2-x^2}{a^2+x^2} \times \frac{a^4-x^4}{(a+x)^3}$.

25. $\frac{a^4-x^4}{a^3-x^3} \times \frac{a^2-x^2}{a^3+x^3} \times \frac{a^4+a^2x^2+x^4}{a^2+x^2}$.

33. Multiplication and division of fractions.

Simplify the following:

1. $\frac{18xy}{25mn} \div \frac{6ab}{35rs}$.

2. $\frac{14a^2b^3c}{15x^2y^5z^6} \div \frac{35x^7y^4z^8}{9a^4b^5c^2}$.

3. $\frac{m+n}{m-n} \div \frac{n+m}{n-m}$.

4. $\frac{x^2-9}{x^2+x} \div \frac{x-3}{x^2-1}$.

5. $\frac{6ab-6b^2}{a(a+b)} \div \frac{2b^2}{a(a^2-b^2)}$.

6. $\frac{x^2-x-6}{x^2+x-6} \div \frac{x^2+4x+4}{x^2+2x-3}$.

7. $\frac{4x^2-1}{6x^2-13x+6} \div \frac{2x^2+3x+1}{2x^2-x-3}$.

8. $\frac{x^2-3x+2}{x^2-7x+12} \div \frac{x^2+x-2}{x^2-x-6}$.

9. $\left(3x-5-\frac{2}{x}\right)\left(3x+5-\frac{2}{x}\right) \div \left(x-\frac{4}{x}\right)$.

10. $\frac{xy-2y}{x^2-x} \times \frac{x^2+xy}{xy-y^2} \div \frac{xy+y^2}{x^2-xy}$.

11. $\frac{x^2+(a+c)x+ac}{x^2+(b+c)x+bc} \div \frac{x^2-a^2}{x^2-b^2}$.

12. $\dfrac{x^2-5x+6}{x^2+x-2}\times\dfrac{x^2+3x-4}{ax+bx-3a-3b}\div\dfrac{x^2+2x-8}{ax-a+bx-b}$.

13. $\left(\dfrac{x^4-a^4}{x^2-2ax+a^2}\div\dfrac{x^2+ax}{x-a}\right)\times\dfrac{x^5-a^2x^3}{x^3+a^3}\div\left(\dfrac{x}{a}-\dfrac{a}{x}\right)$.

14. $\left(\dfrac{x+y}{y}+\dfrac{y}{x+y}\right)\div\dfrac{x+y}{xy}$.

15. $\dfrac{a-1}{2}\times\dfrac{1+a+a^2}{a}\div\dfrac{a^3-1}{a^2}$.

16. $\dfrac{(a+b)^2-4ab}{(a-b)^2+4ab}\div\dfrac{a^2-2ab+b^2}{a^2-b^2}$.

17. $\left(2-\dfrac{a+b}{a-b}\right)\div\left(2+\dfrac{a+b}{a-b}\right)$.

18. $\dfrac{a^3-b^3}{a^2-b^2}\times\dfrac{a^4-b^4}{a-b}\div\dfrac{a^2+b^2}{a^3+b^3}$.

19. $\dfrac{2(a-x)}{a(a^2+2a)}\times\dfrac{a+b}{a-b}\div\dfrac{a^2+ab-ax-bx}{a^2+2a-ab-2b}$.

20. $\left(\dfrac{2}{x}-\dfrac{1}{a+x}+\dfrac{1}{a-x}\right)\div\left(\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}\right)$.

34. Complex fractions.

Simplify the following:

1. $\dfrac{\frac{a}{2}-\frac{b}{3}}{\frac{a}{2}+\frac{b}{3}}$.

2. $\dfrac{1-\frac{y^2}{x^2}}{1+\frac{y^2}{x^2}}$.

3. $\dfrac{\frac{1}{x}-\frac{2}{x^2}-\frac{3}{x^3}}{\frac{9}{x}-x}$.

4. $\dfrac{\frac{a}{x^2}+\frac{x}{a^2}}{\frac{1}{a^2}-\frac{1}{ax}+\frac{1}{x^2}}$.

5. $\dfrac{2x + \dfrac{x}{x-2}}{2x - \dfrac{x}{x-2}}$.

6. $\dfrac{\dfrac{4x^2 - 16x + 15}{(2x-3)^2}}{\dfrac{4}{} - 1}$.

7. $\dfrac{\dfrac{1}{2x-1} - \dfrac{1}{2x+1}}{4 - \dfrac{1}{x^2}}$.

8. $\dfrac{\dfrac{x^2 + 10x + 21}{x^2 - 2x - 15}}{\dfrac{2x^2 + x - 15}{2x^2 - 19x + 35}}$.

9. $\dfrac{\dfrac{a^3 + b^3}{a^2 - b^2}}{\dfrac{a^2 + ab}{a^2 - ab}}$.

10. $\dfrac{2 + \left(\dfrac{a-x}{a+x}\right)^2}{2 - \left(\dfrac{a-x}{a+x}\right)^2}$

11. $\dfrac{\dfrac{12x^2 + x - 1}{15x^2 + 8x + 1}}{\dfrac{4x^2 - 5x + 1}{12x^3 - 12x^2 + 4x - 4}}$.

12. $\dfrac{\dfrac{x+y}{x-y} - \dfrac{x-y}{x+y}}{1 - \dfrac{x^2 + y^2}{(x+y)^2}}$

13. $\dfrac{\dfrac{1}{x^2 - 9x + 20} + \dfrac{1}{x^2 - 11x + 30}}{\dfrac{2}{x^2 - 10x + 24}}$.

14. $\dfrac{\dfrac{1}{x^2 - 7x + 12} - \dfrac{1}{x^2 - 5x + 6}}{\dfrac{x-1}{x^2 - 7x + 12}}$.

15. $\dfrac{\dfrac{(xy - 3y^2)^2}{x^3y^2 - 27y^5}}{\dfrac{x^2 + xy - 2y^2}{x^3 - y^3}} \times \dfrac{x + 2y}{(x - 3y)(x^3 - y^3)}$.

16. $\dfrac{\dfrac{x^2 - y^2}{2} - \left(\dfrac{1}{2x+1} + \dfrac{1}{2x-1} + \dfrac{4x}{1 - 4x^2}\right)}{\dfrac{x+y}{x-y}}$.

17. $\dfrac{2+\dfrac{x^2-5x+6}{x^2-7x+12}}{3-\dfrac{x^2-9x+20}{x^2+6x-55}}$.

18. $\dfrac{\dfrac{(a^3-b^3)(a+b)}{(a^3+b^3)(a-b)}}{\dfrac{a^4-b^4}{(a^3-b^3)(a+b)}} \times \dfrac{a^2-ab+b^2}{a^2+ab+b^2}$.

19. $\dfrac{\dfrac{xy-y^2}{x^2-xy}+\dfrac{x^2-y^2}{xy}}{\dfrac{x(x+y)}{y(x-y)}}$.

20. $\dfrac{\dfrac{2ax}{x^2-a^2}+\dfrac{3a}{a+x}+\dfrac{a}{a-x}}{\dfrac{a-x}{2a^2+3ax+x^2}} \times \dfrac{a^2-x^2}{4a}$.

21. $\dfrac{\left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}-\dfrac{4y^2}{x^2-y^2}\right)\left(\dfrac{x+y}{2y}\right)}{\dfrac{x^2-y^2}{x^2+xy-2y^2} \div \dfrac{x+y}{x-y}}$.

22. $\dfrac{1+\dfrac{1}{x^2}+\dfrac{1}{x^4}}{1+\dfrac{1}{x}+\dfrac{1}{x^2}}$.

23. $\dfrac{\left(x+2+\dfrac{4}{x-3}\right)\left(x-2-\dfrac{4}{x+1}\right)}{\dfrac{x^2-x-2}{x^2-5x-6}}$.

24. $\dfrac{\left(1+\dfrac{a}{b}\right)\left(1-\dfrac{a}{b}\right) \div \dfrac{a-b}{a(a+b)}}{\dfrac{a^2-b^2}{a-b} \times \dfrac{a+b}{b-a}}$.

25. $\dfrac{\dfrac{x^4 - 2x^2y^2 + y^4}{x^3y + xy^3} \div \dfrac{x^2 - y^2}{x^2 + y^2}}{\dfrac{x^2 - y^2}{4x^2y^2}}$.

35. Miscellaneous examples in fractions.

From College Entrance Examinations

Simplify:

1. $\dfrac{a^3 - c^3}{a - c} + a\left[\frac{1}{2} + \dfrac{2c^2 - \frac{1}{2}}{2c + 1}\right]$.

2. $\left[\dfrac{a^2 + ax}{\dfrac{2x}{a^2 - x^2}}\right]\left[\dfrac{(a + x)^2}{4ax} - 1\right]$.

3. $\dfrac{a^2b^2 - a^2 - b^2 + 1}{ab - a - b + 1}$.

4. $\dfrac{2}{x} - \dfrac{3}{2x + 1} + \dfrac{2x - 3}{1 - 4x^2}$.

5. $\dfrac{8c^3 - 1}{9c^2 - 12c + 4} \cdot \left(1 - \dfrac{4}{3c + 2}\right) \div \left(\dfrac{2c - 1}{9c^2 - 4}\right)$.

6. $\dfrac{\left(y^2 + \dfrac{16}{y^2 - 4}\right)\left(y^2 + 4\right)}{\dfrac{y}{y + 2} + \dfrac{2}{y - 2}}$.

7. $\dfrac{1}{p + 1} - \dfrac{1}{1 - p} - \dfrac{2}{p - 2} + \dfrac{2}{p + 2}$.

8. $\dfrac{1}{1 + \dfrac{x}{1 + x + \dfrac{2x^2}{1 - x}}}$.

9. $\left[\dfrac{x + 4}{x^2 - x - 12} - \dfrac{x + 3}{x^2 + x - 12}\right] \div \left[1 + \dfrac{2(x^2 - 12)}{x^2 + 7x + 12}\right]$

10. $\dfrac{\dfrac{2y}{x+2y} - \dfrac{x}{2y-x} + \dfrac{8y^2}{x^2-4y^2}}{\dfrac{4y-x}{(2y-x)^2}}$.

Review Exercises from College Entrance Examinations

1. Determine the H. C. F. and L. C. M. of $(xy - y^2)^3$ and $y^3 - x^2y$.
2. Find the value of $a - \{5b - [a - (3c - 3b) + 2c - 3(a - 2b - c)]\}$ when $a = -3; b = 4; c = -5$.
3. Factor: (1) $x^2 - 4ax - 4b^2 + 8ab$; (2) $(a + b)(c^2 - d^2) - (a^2 - b^2)(c - d)$.
4. Simplify: $\left[x - \dfrac{y(x-y)}{x+y}\right] \cdot \left[x - \dfrac{y^2(x-y)}{x^2+y^2}\right] \div \left[1 - \dfrac{xy-y^2}{x^2}\right]$.
5. Find the H. C. F. and L. C. M. of : $m^4 - n^4$; $m^3 + m^2n - mn^2 - n^3$; $m^4 - 2m^2n^2 + n^4$.
6. Simplify: $\dfrac{\dfrac{a+b}{ab}\left(\dfrac{1}{a} - \dfrac{1}{b}\right) - \dfrac{b+c}{bc}\left(\dfrac{1}{c} - \dfrac{1}{b}\right)}{\dfrac{a+c}{ac}\left(\dfrac{1}{a} - \dfrac{1}{c}\right)}$.
7. Find the H. C. F. and L. C. M. of: $10ab^2(x^2 - 2ax)$; $15a^3b(x^2 - ax - 2a^2)$; $25b^3(x^2 - a^2)^2$.
8. Simplify: $\dfrac{2a-3}{1-4a^2} - \dfrac{3}{1-2a} + \dfrac{2(2a-1)}{a+2a^2}$.
9. Simplify: $\left(5 - \dfrac{a^2 - 19x^2}{a^2 - 4x^2}\right) \div \left(3 - \dfrac{a-5x}{a-2x}\right)$
10. Factor: $36a^2 - 24 + 5a$; $2x^5 - 32x$; $\dfrac{8x^3}{27y^3} - \dfrac{y^3}{x^3}$.
11. Simplify: $\left(1 - \dfrac{1-x}{1+x} + \dfrac{1+2x^2}{1-x^2}\right)\left(\dfrac{x+1}{2x+1}\right)$.
12. Simplify: $\dfrac{a^3 - x^3}{a^2 + 2ax + x^2} \cdot \dfrac{5x}{3x - 3a} \cdot \dfrac{6a + 6x}{5a^2 + 5ax + 5x^2}$.

13. Simplify: $\dfrac{xy}{x^2+y^2}\left(\dfrac{x+y}{x-y}+\dfrac{x^3+y^3}{x^3-y^3}\right) \div \left(\dfrac{x+y}{x-y}-\dfrac{x^3+y^3}{x^3-y^3}\right)$.

14. Simplify: $2+\dfrac{1}{x}-\left[2x-3\left(\dfrac{1}{x+1}+\dfrac{1}{x-1}\right)-\left(\dfrac{1}{x-1}-\dfrac{1}{x}-2\right)\right]$ and check by placing $x=3$.

15. Simplify: $\dfrac{1-\dfrac{2xy}{(x+y)^2}}{1+\dfrac{2xy}{(x-y)^2}} \div \left\{\dfrac{1-\dfrac{y}{x}}{1+\dfrac{y}{x}}\right\}^2$.

XI. LINEAR EQUATIONS

36. Numerical equations.

Solve the following equations:

1. $\frac{x}{2} - \frac{4x}{9} = 5.$

2. $\frac{x}{2} + \frac{x}{3} - \frac{x}{4} = 7.$

3. $3y + \frac{1}{2}(5y - 3) = 20\frac{1}{2}.$

4. $\frac{5r + 1}{2r + 7} = 1.$

5. $\frac{4x - 3}{2x - 1} = \frac{4x - 7}{2x - 5}.$

6. $\frac{.25x - 1.5}{.15(x - 5)} = \frac{2}{15}.$

7. $\frac{2(x + 1.5)}{5(.8x - 1)} = \frac{15}{19}.$

8. $\frac{6x + 7}{5} - \frac{2x - 1}{10} = 4\frac{1}{2}.$

9. $\frac{8x - 9}{14} - \frac{3x - 5}{21} + \frac{2x - 7}{3} = 0.$

10. $\frac{2x - 5}{9} - \frac{x + 3}{5} - \frac{x}{3} = 2x - 17\frac{1}{3}.$

11. $\frac{8x + 37}{18} - \frac{7x - 29}{5x - 12} = \frac{4x + 12}{9}.$

12. $\frac{7}{2x + 6} - \frac{3\frac{1}{2}}{2x + 2} = \frac{4}{x + 3} - \frac{2}{x + 1}.$

13. $\frac{8x + 5}{14} - \frac{3 - 7x}{6x + 2} = \frac{16x + 15}{28} + \frac{2\frac{1}{4}}{7}.$

14. $\dfrac{6x+7}{15} - \dfrac{2x-2}{7x-6} = \dfrac{2x+1}{5}$.

15. $(3x-1)^2 + (4x-2)^2 = (5x-3)^2$.

16. $\dfrac{5x^2+x-3}{5x-4} = \dfrac{7x^2-3x-9}{7x-10}$.

17. $\dfrac{2}{2x+1} - \dfrac{3}{3x-1} = \dfrac{3}{3x-2} - \dfrac{2}{2x-3}$.

18. $\dfrac{x}{x-2} - \dfrac{x+1}{x-1} = \dfrac{x-8}{x-6} - \dfrac{x-9}{x-7}$.

19. $\dfrac{5}{x-2} - \dfrac{10}{2x+4} = \dfrac{2}{x-4} - \dfrac{2}{x+6}$.

20. $\dfrac{x+5}{x+4} + \dfrac{x-15}{x-16} = \dfrac{x-4}{x-5} + \dfrac{x-6}{x-7}$.

21. $\dfrac{x(x+3)}{(x+1)(x+2)} + \dfrac{4}{3x(x+2)} = 1$.

22. $\dfrac{4x+15}{6} + \dfrac{\dfrac{2x}{5} - \dfrac{1}{2}}{6} - \dfrac{x+\frac{1}{4}}{\frac{1}{5}} = 5\frac{13}{30}$.

23. $\dfrac{3(x+2)}{\frac{4}{3}} + \dfrac{\frac{1}{5}(6x-2)}{\frac{1}{7}} = \dfrac{\dfrac{2x}{3}+1}{\frac{1}{6}} + 9$.

24. $\dfrac{3\frac{1}{4}-4x}{1\frac{1}{3}} - \dfrac{49}{54}(3\frac{1}{2}-5x) = \dfrac{7}{16} + \dfrac{55}{108}(3x-2)$.

25. $\dfrac{2}{1-2x} - \dfrac{2}{2x-7} = 1 - \dfrac{4x^2-1}{4x^2-16x+7}$.

26. $\dfrac{x}{2} - \dfrac{\frac{1}{3}(2x-3) - \frac{1}{4}(3x-1)}{\frac{1}{2}(x-1)} = \dfrac{3}{2}\left(\dfrac{x^2+2}{3x-1}\right)$.

27. $\dfrac{x+2}{x} + \dfrac{x-7}{x-5} - \dfrac{x+3}{x+1} = \dfrac{x-6}{x-4}$.

28. $\dfrac{\frac{1}{3}(x-2)}{\frac{2}{3}} + \dfrac{\frac{3}{4}(6x-7)}{5} = \dfrac{\dfrac{2x}{3}-1}{\frac{1}{6}} - 1\frac{1}{4}$.

29. $\dfrac{1}{x-2} - \dfrac{1}{x-4} = \dfrac{1}{x-6} - \dfrac{1}{x-8}$.

30. $4x - \{3x - (2x - \overline{6x+1})\} = 4x - (5x - 2) - 23$.

31. $3[4x - 5 - 2(3x - 4) + 5\{2x - 3 - (2x - 7[x - 5])\}] = 39 - x$.

32. $7x^2 - [2x - \{2x^2 - (3x - 4)\} + 3x] = 9x^2 + 35 - [5x - (3x + 7)]$.

37. Literal equations.

Solve the following equations:

1. $\dfrac{x}{a} = b$.

2. $\dfrac{y}{3} = \dfrac{a}{6}$.

3. $\dfrac{ay}{b} = 1$.

4. $\dfrac{3x}{a} + \dfrac{2x}{b} = 2$.

5. $\dfrac{z-a}{b} = \dfrac{z-b}{a}$.

6. $\dfrac{c+d}{x} - a = \dfrac{c-d}{x} + b$.

7. $\dfrac{m-z}{n-z} = \dfrac{m+z}{n+z}$.

8. $\dfrac{6y+a}{4y+b} - \dfrac{3y-b}{2y-a} = 0$.

9. $\dfrac{x-a}{x+a} = \dfrac{3b+x}{2b+x}$.

10. $\dfrac{x-a}{a-b} - \dfrac{x+a}{a+b} = \dfrac{2ax}{a^2-b^2}$.

11. $\dfrac{1}{a} - \dfrac{1}{x} = \dfrac{1}{x} - \dfrac{1}{b}$.

12. $\dfrac{x}{x-a} - \dfrac{x+a-b}{x-b} = \dfrac{a(a-b)}{(x-c)(x-d)}$.

13. $a^2(x-a) + b^2(x-b) = abx$.

14. $\dfrac{x-1}{x+1} = \dfrac{1-a}{1+a}$.

15. $(x+2a)(x-a)^2 = (x+2b)(x-b)^2$.

16. $(x-a)(x-b) = (x-a-b)^2$.

17. $\dfrac{a}{x-a} - \dfrac{b}{x-b} = \dfrac{a-b}{x-c}$.

18. $\dfrac{a}{x+a} + \dfrac{b}{x+b} = \dfrac{a+b}{x+c}$.

19. $\dfrac{x^2-a^2}{bx} - \dfrac{a-x}{b} = \dfrac{2x}{b} + \dfrac{a}{x}$.

20. $x(x-a) + x(x-b) = 2(x-a)(x-b)$.

21. $(x-a)^2 - (x-b)^2 = (a-b)^2$.

22. $\dfrac{x}{a} + \dfrac{x}{b-a} = \dfrac{a}{b+a}$.

23. $\dfrac{a+b}{x} + \dfrac{a^2-b^2}{a^2+ab} = \dfrac{a}{x} - \dfrac{b}{a}$.

24. $(a-b)(x-c) - (b-c)(x-a) - (c-a)(x-b) = 0$.

25. $\dfrac{a^2c}{b} + x^2 = (a+x)(b+x) - a(b+c)$.

26. $\dfrac{x+a}{x-a} - \dfrac{x-a}{x+a} - \dfrac{x^2}{a^2-x^2} = 1$.

27. $\dfrac{x+7a}{x+6a} + \dfrac{x-a}{x-3a} + \dfrac{a-x}{x+2a} = \dfrac{x+7a}{x+a}$.

28. $\dfrac{x^3+2a^3}{x+a} - \dfrac{x^3-2a^3}{x-a} = \dfrac{a^3}{x^2-a^2} - 2ax$.

38. Formulas.

1. Solve for R: $S = \pi RL$.
2. Solve for L: $T = \pi R(R+L)$.

3. Solve for T: $D = RT$.

4. Solve for F: $C = \frac{5}{9}(F - 32)$.

5. Solve for s: $R = \dfrac{gs}{g + s}$.

6. Solve for n: $l = a + (n - 1)d$.

7. Solve for l: $S = \dfrac{n}{2}(a + l)$.

8. Solve for d: $S = \dfrac{n}{2}[2a + (n - 1)d]$.

9. Solve for b': $A = \frac{1}{2}h(b + b')$.

10. Solve for r: $a = p + prt$.

11. Solve for r: $S = \dfrac{rl - a}{r - 1}$.

12. Solve for t_1: $V_1 = V_0(1 + .00365t_1)$.

39. Problems producing linear equations of one unknown quantity.

1. The sum of two numbers is 52, and the greater exceeds the less by 4. What are the numbers?

2. The difference of two numbers is 5, and the difference of their squares is 55. What are the numbers?

3. The difference of two numbers is 36, and the greater is ten times the less. What are the numbers?

4. The sum of two numbers is 90, and $\frac{1}{5}$ of the greater is equal to $\frac{1}{4}$ of the less. What are the numbers?

5. Find a number which is as much above 50 as $\frac{1}{6}$ of the same number is below 20.

6. Ten times the difference between the fifth and sixth parts of a certain number is 40 less than the number. What is the number?

7. Divide 100 into two such parts that four times the greater part shall exceed five times the less by 22.

8. A grocer has two kinds of coffee, one worth 20 cents a pound, the other worth 30. He wishes to make 100 pounds of a mixture worth 26 cents a pound. How many pounds of each sort should he take?

9. Ten dollars were distributed among 5 men, 3 women, and 14 children, so that a man received as much as 2 women, and a woman as much as 2 children. How much did each child receive?

10. Into a cask $\frac{2}{3}$ full 10 gallons were poured and it was found to be $\frac{5}{6}$ full. How many gallons did the cask hold?

11. A man has \$8000 which he wishes to invest in two enterprises so that his total income will be \$425; if one enterprise pays $5\frac{1}{2}\%$ and the other 5%, how much must he invest in each?

12. Of 24 pounds of salt water, 12% is salt; of another mixture 4% is salt. How many pounds of the second should be added to the first in order to get a mixture 10% salt?

13. Divide \$600 among 5 persons so that the first and second together shall receive \$150; the first and third, \$250; the first and fourth, \$200; and the fifth \$100 less than the third. How much will each receive?

14. Into what two sums can \$1000 be divided so that the income of one at 6 per cent shall be equal to the income of the other at 4 per cent?

15. A man walked from A to B at the rate of 3 miles an hour and rode back at the rate of $9\frac{3}{4}$ miles an hour, being gone $10\frac{1}{2}$ hours. How far is it from A to B?

16. A boy buys a certain number of apples at the rate of 3 for 7 cents and $\frac{2}{3}$ as many more at the rate of 2 for 5 cents. If he sells the whole number at the rate of 2 for 9 cents and gains 84 cents, how many does he buy?

17. A marksman hears the bullet strike the target 3 seconds after the report of his rifle. If the bullet travels

1925 feet per second and sound travels 1100 feet per second, find the distance of the target and the time the bullet was in the air.

18. A company of 500 soldiers has provisions for 60 days. After 12 days it is joined by another detachment and the provisions last only 40 days longer. How many men in the new detachment?

19. A boy is $\frac{1}{6}$ as old as his father and 5 years older than his sister. If the sum of the ages of all three is 51, how old is the father?

20. A man has 2 hours at his disposal. How far can he ride at the rate of 10 miles an hour and return by walking at the rate of 3 miles an hour?

21. Two trains leave a town at the same time and travel in the same direction; the faster one runs m miles per hour and the slower runs s miles per hour. In how many hours will they be k miles apart?

22. Two-thirds of a man's property is invested at 5 per cent, $\frac{1}{6}$ at 4 per cent, and the remainder at 6 per cent. If the total income is \$450, what is the value of the property?

23. The distance from A to B is 100 miles. A train leaving A at a certain rate meets with an accident 20 miles from B, reducing its speed one-half, and causing it to reach B an hour late. What was its regular rate per hour?

24. Two trains start from the same station going in the same direction, one at the rate of 20 miles per hour, and the other 40 minutes later at the rate of 30 miles per hour. If the faster train be delayed 30 minutes by an accident, at what distance from the starting-point will it overtake the other?

25. A and B trade with equal capital. In the first year A doubled his capital and had \$500 over, but in the same time B lost \$500 less than $\frac{1}{2}$ of his capital. If A's gain was five times B's loss, what was the capital of each?

26. A certain number consists of two digits of which the one in units' place is four times the one in tens' place. If the order of the digits is inverted and 2 is added to the resulting number, the new number will be three times the original number. Find the original number.

27. A grain dealer invests \$9000 in wheat. He sells one fourth of the wheat at an advance of 6 cents per bushel and the remainder at an advance of 10 cents per bushel and clears \$900. What was the original price?

28. The cost of publication of each copy of a certain magazine is $6\frac{1}{2}$ cents. It sells to dealers for 6 cents and the amount received for advertising is 10% of the amount received for all magazines issued beyond 10,000. Find the least number of magazines which can be issued without loss.

29. A number consists of three digits of which the second is 2 greater than the first, and the third, equal to the sum of the first two. If the first and third change places, the new number will be 297 more than the original number. Find the original number.

30. A can do a piece of work in 10 days, and B can do the same work in 15 days. A works alone a number of days, after which B finishes it alone in $12\frac{1}{2}$ days. How long does A work?

31. A can do a piece of work in 20 days, and B can do the same work in 30 days. They work together for a time, and then A finishes it alone in 5 days. How many days do they work together?

32. A detachment of men has a supply of bread for 30 days, allowing each man 14 ounces per day. After 15 days 200 men are withdrawn and the allowance per man is increased to 20 ounces per day, which exhausts the supply in $11\frac{2}{3}$ days. How many men were there at first?

33. A man engaged to work eight months for \$200 and a suit of clothes of a certain value. At the end of six months he was discharged for bad conduct and made to forfeit one month's pay. If he received \$113 and the suit of clothes, what was the value of the suit?

34. A and B engaged in trade, A with \$250 and B with \$300. A lost $\frac{3}{4}$ as much as B lost, and B had $\frac{5}{6}$ as much as A remaining. How much did each lose?

35. A agrees to do a certain piece of work for \$500. After working 10 days alone he employs B and C, and the three together finish it in $39\frac{3}{7}$ days. If B can do half as much as A, and C $\frac{2}{3}$ as much as B, what are the daily wages of each?

36. A has \$1200 a year more income than B and each saves $\frac{1}{6}$ of his income. If at the end of 5 years A has saved $\frac{7}{8}$ as much as B, what is the income of each?

37. A man spends $\frac{1}{3}$ of his money and then earns \$1000. He now spends $\frac{1}{4}$ of all he has and afterwards earns \$1500. If the amount which he now has will yield at 3% an annual income of \$165, how much had he at first?

38. A box of oranges was bought at the rate of 15 cents a dozen. Five dozen were given away and the remainder sold at the rate of 2 for 5 cents. If this gave a profit of 30 cents on the box, how many were there in the box?

39. A sets out on a journey at the rate of m miles per hour; and c hours after, B starts after him at the rate of n miles per hour. In how many hours will B overtake A, and how far will each have walked.

40. After giving away half a dollar less than half my money, and a third of a dollar more than a third of what remained, and then a fourth of a dollar less than a fourth of what still remained, I had \$4. How much had I at first?

41. A man starts from his home to catch a train, walking at the rate of one yard in one second, and arrives two minutes late. If he had walked at the rate of four yards in three seconds he would have arrived two and a half minutes early. Find the distance from his home to the station.

42. A number consists of four digits of which the left-hand one is 4. If this digit is placed at the right of the others, the new number will be 1062 larger than the original number. Find the original number.

43. A number consists of three digits. The one in units' place is 5. If this digit is placed at the left of the others and the new number is divided by the original one, the quotient will be 4 and the remainder 12. Find the original number.

44. A company of men was drawn up in a hollow square 4 deep. Afterwards it was separated into two detachments. One was a hollow square 3 deep and with 34 more men in front than at first, and the other contained 80 men. How many men in the original company?

45. A man walks along a railway at the rate of 4 miles an hour. If a train 208 yards long and travelling 30 miles an hour overtakes him, how long will it take in passing the man?

46. One cask contains a mixture of 14 gallons of wine and 16 gallons of water; another contains a mixture of 20 gallons of wine and 10 gallons of water. How many gallons must be drawn from each to make a mixture of 5 gallons of wine and 5 gallons of water?

47. A man rowed down a stream 18 miles in 3 hours, but upon returning found that it took him 3 hours to reach a place 6 miles below his starting-point. Find the rate per hour of the rowing.

48. A man rowed down a stream 11 miles in $1\frac{1}{2}$ hours, but found that it took him $3\frac{3}{4}$ hours to return the same distance. Find the rate per hour of the stream.

49. A boatman can row 6 miles in 4 hours against a certain tide. If the tide were as swift again, it would gain upon him $1\frac{1}{2}$ miles in an hour. Find the rate per hour of the tide.

50. A passenger train traveling m miles an hour starts t hours later than a freight train whose rate is r miles an hour. In how many hours will the passenger train overtake the freight train?

51. A dinner was ordered for 20 persons at such a rate as to gain 25% on the cost; but 5 of them being absent, and the others paying only the original rate, there was a loss of \$1.25. What was the rate for each person?

52. A regiment can be drawn up in two hollow squares, one three deep and the other five deep. If the front line in the first contains 6 men more than the front line in the second, and both squares contain the same number of men, how many men in the regiment?

53. A man has \$25,000. He invests a portion poorly and gets no income from it. One-fifth of the remainder pays him 4%; three-fourths of what still remains, 6%; and the other fourth, 5%. If his income is \$1080 per year, what is the amount of the poor investment?

54. At what time between 5 and 6 will the minute hand of a clock be 8 minute spaces in advance of the hour hand?

55. At what time between 11 and 12 will the hands of a clock be at right angles to each other?

56. The sum of three numbers is 85. If the third is divided by the first, the quotient will be 2 with a remainder of 2; but if the third is divided by the second, the quotient will be 1 with a remainder of 19. What are the numbers?

57. A and B start at the same time to walk from C to D, at the rate of $3\frac{1}{3}$ and 4 miles per hour respectively. After

B has gone $\frac{2}{3}$ of the distance he rests $\frac{3}{5}$ of an hour, which allows A to overtake him. Find the distance from C to D.

58. A man bequeathed his property as follows: To his wife \$1200 and $\frac{1}{5}$ of what remained; to his eldest son $\frac{1}{2}$ as much and $\frac{1}{5}$ of what then remained; and to his second son $\frac{1}{2}$ as much as to the eldest son, and $\frac{1}{2}$ of what still remained. If \$3776 yet remained, what was the whole estate?

40. Numerical simultaneous equations.

Solve the following equations:

1. $\begin{cases} 2x - 3y = -3. \\ 3x - 2y = 8. \end{cases}$

2. $\begin{cases} 5x + 3y = 68. \\ 3x - 2y = 18. \end{cases}$

3. $\begin{cases} 16x - 16y - 1 = \frac{5}{3}. \\ 2x + y + 6 = \frac{2}{3}\frac{2}{8}. \end{cases}$

4. $\begin{cases} 2y - 3x = 6. \\ 2x - 3y = -24. \end{cases}$

5. $\begin{cases} x - \dfrac{5y + 2}{2} = -4. \\ y - \dfrac{2x - 3}{7} = 3. \end{cases}$

6. $\begin{cases} \dfrac{8x - 3}{4} + \dfrac{y - 5}{3} = \dfrac{7}{6}. \\ x - \dfrac{y - 7}{5} = \dfrac{5}{8}. \end{cases}$

7. $\begin{cases} \dfrac{\frac{3x}{2} - \frac{2y}{3}}{1\frac{1}{2}} - \dfrac{4y - 3x}{7} = -\dfrac{4}{3}. \\ y + x = 7(y - x). \end{cases}$

8. $\begin{cases} \dfrac{\frac{2x}{3} - \frac{5y}{12}}{\frac{7}{4}} - \dfrac{\frac{3x}{2} - \frac{y}{3}}{\frac{2}{2}\frac{8}{3}} = 2. \\ 2x - y = 4(2y - x). \end{cases}$

9. $\begin{cases} .3x + .2y = 9.5. \\ .2x + .3y = 10.5. \end{cases}$

10. $\begin{cases} \dfrac{x - y}{2} - \dfrac{x - 3y}{5} = y - 3. \\ \frac{3}{4}(x - y) + \frac{5}{6}(x + y) = 18. \end{cases}$

11. $\begin{cases} \dfrac{9}{2x+3y+6} = \dfrac{17}{9x+2y+3}. \\ \dfrac{5x-7y+2}{9} = -\dfrac{2x+5y-17}{3}. \end{cases}$

12. $\begin{cases} \dfrac{12}{x} + \dfrac{8}{y} = 8. \\ \dfrac{27}{x} - \dfrac{12}{y} = 3. \end{cases}$

13. $\begin{cases} \dfrac{15}{x} - \dfrac{10}{y} = 3. \\ \dfrac{9}{x} + \dfrac{16}{y} = 6\frac{1}{2}. \end{cases}$

14. $\begin{cases} \dfrac{5}{x} + \dfrac{3}{y} = \dfrac{7}{20}. \\ 15x + 10y = xy. \end{cases}$

15. $\begin{cases} 2y - x = 4xy. \\ \dfrac{4}{y} - \dfrac{3}{x} = 9. \end{cases}$

16. $\begin{cases} \dfrac{1}{x} - \dfrac{1}{y} = 1. \\ 2x + y = 7xy. \end{cases}$

17. $\begin{cases} 2x + 3y = 5. \\ \dfrac{5}{x} + \dfrac{7}{y} = \dfrac{12}{xy}. \end{cases}$

18. $\begin{cases} \dfrac{10}{3x} - \dfrac{5}{2y} = \dfrac{5}{6}. \\ \dfrac{10}{x} - \dfrac{5}{y} = \dfrac{10}{3}. \end{cases}$

19. $\begin{cases} \dfrac{9.8}{7x} + \dfrac{10.5}{5y} = 2. \\ \dfrac{29.4}{7x} - \dfrac{6.3}{y} = 1. \end{cases}$

41. Literal Simultaneous Equations:

Solve the following equations:

1. $\begin{cases} ax + by = c. \\ bx - ay = d. \end{cases}$

2. $\begin{cases} x + y = a + b. \\ bx + ay = 2ab. \end{cases}$

3. $\begin{cases} \dfrac{x}{b} + \dfrac{y}{c} = 1. \\ \dfrac{ax}{c} - \dfrac{by}{a} = 0. \end{cases}$

4. $\begin{cases} \dfrac{x}{c} + \dfrac{y}{d} = 1. \\ x - y = c - d. \end{cases}$

5. $\begin{cases} \dfrac{ax}{c} + \dfrac{by}{d} = a + b. \\ \dfrac{x}{3c} + \dfrac{y}{6d} = \dfrac{1}{2}. \end{cases}$

6. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{a}. \\ \dfrac{1}{x} - \dfrac{1}{y} = \dfrac{1}{b}. \end{cases}$

7. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = c. \\ \dfrac{b}{x} + \dfrac{a}{y} = d. \end{cases}$

8. $\begin{cases} x + y = a + b. \\ \dfrac{x + a}{y + b} = \dfrac{b}{a}. \end{cases}$

9. $\begin{cases} x(a + b) + y(a - b) = 2. \\ ax + by = \dfrac{a^2 + b^2}{a^2 - b^2}. \end{cases}$

10. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = 2. \\ \dfrac{x}{b} + \dfrac{y}{a} = \dfrac{a^2 + b^2}{ab} \end{cases}$

11. $\begin{cases} \dfrac{x + y}{a} + \dfrac{x - y}{b} = 1. \\ \dfrac{x - y}{a} - \dfrac{x + y}{b} = 1. \end{cases}$

12. $\begin{cases} \dfrac{x + 1}{y + 1} = \dfrac{m + n + 1}{m - n + 1}. \\ x(m + n) + y(m + n + 1) = 2m^2 + 2mn + m - n. \end{cases}$

13. $\begin{cases} \dfrac{x}{a + b} + \dfrac{y}{a - b} = 2a. \\ \dfrac{x - y}{4ab} = 1. \end{cases}$

14. $\begin{cases} \dfrac{x}{2a} + \dfrac{y}{3b} = \dfrac{5}{6}. \\ \dfrac{x}{5a} - \dfrac{y}{2b} = \dfrac{8}{5}. \end{cases}$

15. $\begin{cases} \dfrac{x}{2a} + \dfrac{y}{3b} = \dfrac{5}{6}. \\ \dfrac{x}{a} + \dfrac{y}{b} = 2. \end{cases}$

16. $\begin{cases} (a - b)x + (a + b)y = 2a^2 - 2b^2. \\ (a + b)x + (a - b)y = 2(a^2 + b^2). \end{cases}$

42. Problems.

1. If 3 yards of velvet and 12 yards of silk cost \$60 and 4 yards of velvet and 5 yards of silk cost \$58 what is the price by the yard of each material?

2. Eight years ago A was five times as old as B, and in two years he will be three times as old. Find their present ages.

3. Thirty bushels of wheat and 25 bushels of corn cost \$49, and 50 bushels of wheat and 10 bushels of corn cost \$50. Find the price of each per bushel.

4. What fraction equals $\frac{3}{4}$ when 3 is added to both numerator and denominator, and equals $\frac{4}{5}$ when 5 is added in the same way?

5. Find the area of a rectangle from the following data: If 6 inches be added to its length and 6 inches to its breadth, the one becomes $\frac{3}{2}$ the other, and the area is increased by 84 square inches.

6. A grocer sold two boxes of raspberries and three of cherries to one customer for 54 cents and three boxes of raspberries and two of cherries to another for 56 cents. Find the price of each per box.

7. A and B can do a piece of work together in 30 days. After they have both worked 12 days B is called away and A finishes it alone in 24 days. How many days would it take each alone?

8. A and B can do a piece of work together in 12 days. A works 3 days alone, they both work together 5 days, and B finishes it alone in 13 days. How many days would it take each alone?

9. Two sums of money are loaned, the first at 5% and the second at 4%, and both together yield an annual income of \$68. If both sums were invested in 6% bonds costing 125, the income would be \$72. Find the sums of money.

10. A man invests \$5000 in 6% bonds at 94, and in 7% bonds at 104. If the total annual income is \$330, what amount is invested at each rate?

11. A cubic inch of aluminum weighs 0.092 pound and of copper 0.31 pound. Find the percentage of composition by weight of a mixture of the two weighing 0.276 pound to the cubic inch.

12. A certain sum of money placed at simple interest amounted to \$1400 in three years, and to \$1500 in five

years. What was the sum at interest, and what was the rate of interest?

13. A number is composed of two digits whose sum is 7. If their order is inverted, the new number will be 2 more than twice the original number. Find the original number.

14. The first of the two digits of a number is $\frac{1}{3}$ of the second. If their order is inverted and the new number is divided by the original number, the quotient will be 2 and the remainder 10. Find the original number.

15. One pound of coffee and 5 pounds of sugar together cost 70 cents; after the price of sugar has advanced $12\frac{1}{2}\%$ and the price of coffee 20%, 3 pounds of sugar and 2 pounds of coffee together cost 99 cents. Find the price of each per pound.

16. A sum of money, at simple interest, amounted in m years to c dollars, and in n years to d dollars. Find the sum and the rate of interest.

17. Three cities, A, B, and C, are so situated that the distance from A to C by way of B is 50 miles, from A to B by way of C is 70 miles, and from B to C by way of A is 60 miles. How far apart are the cities? (Make a diagram.)

18. \$6000 is invested in bonds of three kinds bearing interest at 4, 5, and 6% respectively. The total income received is \$290, and the interest from the 6% bonds equals that from the 4% bonds. How much is invested at each rate?

19. A and B race from C to D, the distance being 100 miles. When B has a start of 16 miles he is beaten by one hour, but when given 6 hours' start wins by 5 miles. Find the rate per hour of each.

20. A and B run a race of 300 yards. When A has a start of 3 seconds he is beaten by 21 yards; when he has a start of 10 yards he is beaten by $6\frac{2}{3}$ seconds. Find the rate per second of each.

21. A person rows a certain distance down a stream, which runs at the rate of 3 miles an hour, in $3\frac{2}{3}$ hours. In returning it takes him 10 hours to reach a place three miles below his starting-point. Find the distance down the stream and the rate of rowing in still water.

22. A man can row 24 miles down a stream and back to the starting-point in 15 hours. He finds he can row as far in 1 hour going down as in 4 hours going up the stream. Find the rate of the current and of the rowing.

23. A boat's crew row 15 miles with the tide in $\frac{3}{4}$ of an hour; when the tide is twice as swift they row 24 miles against it in 3 hours. Find the rate of rowing and of the swifter tide.

24. A rectangle is of the same area as another which is 4 yards longer and 1 yard narrower; it is also of the same area as a third, which is 4 yards shorter and 2 yards wider. What is its area?

25. A chemist has the same acid in two strengths. Eight quarts of one mixed with 12 quarts of the other gives a mixture 84% pure, and 3 quarts of the first with 2 quarts of the second gives a mixture 86% pure. Find the per cent of purity of each acid.

26. Five dry quarts and 10 liquid quarts equal 15 liters, and 7 dry quarts and 4 liquid quarts equal $11\frac{1}{2}$ liters. How many liters are there then in a dry quart and a liquid quart?

27. A cask contains a mixture of 12 gallons of wine and 4 gallons of water; another cask contains a mixture of 8 gallons of wine and 12 of water. How many gallons must be drawn from each to make a mixture of 7 gallons of wine and 7 of water?

28. A railway train after running 5 hours meets with an accident which delays it 30 minutes, after which it proceeds at $\frac{3}{4}$ of its usual speed, and reaches its destination 3 hours

late. If the accident had happened 40 miles nearer the starting-point, it would have been $3\frac{1}{3}$ hours late. Find the regular speed and the length of the line.

29. An income of \$145 is derived from a sum of money invested partly in $5\frac{1}{2}\%$ and partly in 4% bonds. If the $5\frac{1}{2}\%$ bonds are sold at 110, and the 4% bonds at 120, the amount realized is \$3140. How much is invested in each kind of bond?

30. A boat goes up stream 30 miles and down stream 44 miles in 10 hours. Again it goes up stream 40 miles and down stream 55 miles in 13 hours. Find the rates per hour of the stream and of the boat.

Linear Equations Containing Three or More Unknown Quantities

43. Numerical and literal equations.

Solve the following equations:

1.
$$\begin{cases} 2x + 3y + z = 86. \\ 5x - y - z = 26. \\ 4x - y + 3z = 94. \end{cases}$$

2.
$$\begin{cases} 2x + 3y + 5z = 3. \\ 4x - 6y + z = \frac{1}{5}. \\ x - y - z = -\frac{1}{10}. \end{cases}$$

3.
$$\begin{cases} 4x + 2y = 50. \\ y + z = 6. \\ x + z = 11. \end{cases}$$

4.
$$\begin{cases} \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 8. \\ \frac{x}{4} + \frac{y}{9} - \frac{z}{6} = 0. \\ \frac{x}{3} + \frac{y}{4} + \frac{z}{5} = \frac{359}{60}. \end{cases}$$

5.
$$\begin{cases} \frac{1}{x} + \frac{1}{y} = 8. \\ \frac{1}{y} + \frac{1}{z} = 15. \\ \frac{1}{x} + \frac{1}{z} = 13. \end{cases}$$

6.
$$\begin{cases} \frac{2}{x} + \frac{3}{y} + \frac{1}{z} = \frac{41}{40}. \\ \frac{3}{x} - \frac{2}{y} + \frac{3}{z} = \frac{77}{120}. \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{59}{120}. \end{cases}$$

7.
$$\begin{cases} \frac{1}{2x}+\frac{1}{2y}+\frac{1}{2z}=\frac{13}{24}. \\ \frac{1}{3x}-\frac{1}{5y}+\frac{1}{z}=\frac{7}{20}. \\ \frac{1}{4x}-\frac{1}{y}-\frac{1}{3z}=-\frac{7}{24}. \end{cases}$$

8.
$$\begin{cases} \frac{x}{3+y}=\frac{3}{7}. \\ \frac{y}{4+z}=\frac{4}{9}. \\ \frac{z}{x+5}=\frac{5}{8}. \end{cases}$$

9.
$$\begin{cases} \frac{xy}{x+y}=70. \\ \frac{xz}{x+z}=84. \\ \frac{yz}{y+z}=140. \end{cases}$$

10.
$$\begin{cases} x+y+z+u=36. \\ 4x+y-z=22. \\ 4z-u=28. \\ 2u+y=32. \end{cases}$$

11.
$$\begin{cases} x+y+z=6. \\ y+z+u=9. \\ z+u+x=8. \\ u+x+y=7. \end{cases}$$

12.
$$\begin{cases} 2x+y+z+u=63. \\ x+y+2z+2u=61. \\ 3x+4y+3z+4u=142. \\ x-y+z-3u=1. \end{cases}$$

13.
$$\begin{cases} 7x-3y=1. \\ 11z-7u=1. \\ 5z+6y=134. \\ 2x+u=33. \end{cases}$$

14.
$$\begin{cases} 2x+3z-u=25. \\ 3z+y+2v=33. \\ 6y-x+u=14. \\ 10y+u+3v=43. \\ 2z+u=22. \end{cases}$$

15.
$$\begin{cases} ay+bx=c. \\ cx+az=b. \\ bz+cy=a. \end{cases}$$

16. $\begin{cases} (b-c)x + (c-a)y + (a-b)z = 0. \\ ax + (a-c)y - cz = 0. \\ x + y + z = a + b + c. \end{cases}$

17. $\begin{cases} x + y + z = 0. \\ ax + by + cz = 0. \\ bcx + acy + abz + (b-c)(c-a)(a-b) = 0. \end{cases}$

44. Problems.

1. Three boys have together 80 marbles. If A gives $\frac{1}{2}$ of his to B and C in equal shares, A and C will have the same number, and B as many as both together. How many had each at first?

2. Separate 100 into three parts such that if the second part be divided by the first the quotient is 4 and the remainder 2; and if the third be divided by the second the quotient is 3 and the remainder 7.

3. There is a number consisting of three digits. The sum of the first and second digits exceeds the third by 4. The sum of the three digits is 10; and, when the order of the digits is inverted, the original number is increased by 99. Find the number.

4. A, B, C, and D are situated in the given order upon the same road. The distance between A and B is 6 miles greater than the distance between C and D; the distance between A and C is $\frac{1}{16}$ of a mile less than $\frac{3}{5}$ as great as that between B and D; and the point half way between A and D is between B and C, and $\frac{1}{2}$ a mile from B. Find the distances AB, BC, and CD.

5. A, B, and C can do a piece of work in 20 days; A and B, in 30 days; and B and C, in 40 days. How long would it take each to do the work alone?

6. A, B, and C can do a piece of work in 15 days. A and B together can do $\frac{4}{3}$ as much as C, and C twice as much as A. How long would it take each to do the work alone?

7. A school is divided into three classes. The first and second contain 50 pupils more than half the whole number; the second and third 30 more than half the whole; and the first and third 10 more than half the whole. Find the number in each class.

8. A grain dealer sold to one customer 5 bushels of wheat, 2 of corn, and 3 of rye, for \$6.60; to another, 2 of wheat, 3 of corn, and 5 of rye, for \$5.80; and to another, 3 of wheat, 5 of corn, and 2 of rye, for \$5.60. What was the price per bushel of each kind of grain?

9. The three angles of a triangle are together equal to 180°. The largest angle is 4 times as large as the smallest one, and equal to the sum of the two smaller angles. Find the three angles.

10. A cistern has three pipes, A, B, and C. A and B can fill it in 20 minutes; A and C, in $16\frac{2}{3}$ minutes; and B and C, in $14\frac{2}{7}$ minutes. Find the time required for each alone to fill it.

11. A cistern has three pipes, A, B, and C. If A and B run in while C runs out, it will be filled in $28\frac{4}{7}$ minutes. If B and C run in while A runs out, it will be filled in 40 minutes. If A and C run in while B runs out, it will be filled in $66\frac{2}{3}$ minutes. In what time would each fill it alone, the others not running?

12. In a 10-mile race A can beat B by 2 miles, and C by 4 miles. By how many miles can B beat C?

13. \$1000 is divided among A, B, C, and D. B gets half as much as A; the excess of C's share over D's share is equal to one-third of A's share, and if B's share were increased by \$100 he would have as much as C and D have between them. Find how much each gets.

45. Review equations and problems.

From College Entrance Examinations

1. Solve: $\frac{2x+a}{2x} + \frac{4x}{2x+a} = 3.$

2. Solve for x and y: $\frac{x}{a} + \frac{y}{b} = 1, \frac{x}{b} - \frac{y}{a} = \frac{1}{2}.$

3. Solve for x, y, and z: $3x - y + 2z = 11$, $3y - z + 2x = 9$; $3z - x + 2y = 16$.

4. Determine graphically the values of x and y in $-2x + y = -3$, $-3x + 4y = 8$.

5. From the equation $s = \frac{n}{2}[2a + (n-1)d]$ find the value of a in terms of the other letters. Compute its numerical value when $s = 2$, $n = 10$, $d = -\frac{2}{5}$.

6. Solve: $\frac{2x}{3} - \frac{5y}{12} - \left(\frac{3x}{2} - \frac{4y}{3}\right) = -\frac{2}{3}, \frac{x-y}{x+y} = \frac{1}{5}.$

7. Solve: $$\frac{x - \frac{3(x-5)}{4}}{8} - \frac{x-3}{4} = x + 1 - \frac{8x - \frac{2(x+16)}{17}}{6}.$$

8. A person has five hours at his disposal. How far can he ride in a buggy, going ten miles an hour, and walk back at the rate of four miles an hour?

9. A steamer can run 20 miles an hour in still water. If it can go 72 miles with the current in the same time that it could run 48 miles against the current, what is the speed of the current?

10. Solve for x and y: $\frac{x}{a+b} + \frac{y}{a-b} = 2a, \frac{x-y}{4ab} = 1.$

11. Solve: $\frac{3.7x}{.9} - \frac{2\frac{3}{4}x}{.02} = 1.72 + \frac{x}{.04}$ and find value correct to three decimal places.

12. Solve: $\frac{1}{x-2} - \frac{1}{x-1} = \frac{1}{x-4} - \frac{1}{x-3}$.

13. Solve for x, y, and z: $2x + y = 9$, $4x + 3y + z = 13$, $x - y - 2z = 16$.

14. Solve: $\frac{5}{3x} - \frac{7}{y} = -\frac{29}{9}$; $\frac{3}{x} + \frac{5}{4y} = \frac{9}{8}$.

15. Solve: $\frac{b}{x} + \frac{a}{y} = c$, $\frac{a}{z} + \frac{c}{x} = b$, $\frac{c}{y} + \frac{b}{z} = a$.

16. Find two numbers such that if the first be divided by m and the quotient increased by a the result will be n times the second number, while if the second be multiplied by p and the product decreased by b the result will be the quotient obtained by dividing the first number by q.

17. A, B, and C can together do a piece of work in 8 days. A and B can together do it in 12 days, and A does as much as B and C together. In what time can each do the work?

18. A physician having 100 cu. cm. of a 6% solution of a certain kind of medicine wishes to dilute it to a $3\frac{1}{2}$% solution. How much water must he add? (A 6% solution contains 6% of medicine and 94% of water.)

19. A man has an appointment which he can just keep if he drives his car 25 miles an hour. Being delayed 1 hour in starting he drives 30 miles an hour, but arrives 10 minutes late. At what rate should he have driven to keep his appointment?

20. A person who possesses \$15,000 employs a part of the money in building a house. He invests one third of the money which remains at 6%, and the other two thirds at 9%, and from these investments he obtains an annual income of \$500. What was the cost of the house?

21. (*a*) Find the value of t from the equation $v = u + ft$; substitute this value in the equation $s = ut + \frac{1}{2}ft^2$ and simplify the result.

(*b*) Solve for s the equation obtained in (*a*) when $f = 32$, $v = 5.1$, and $u = 1.3$.

22. A merchant mixes two kinds of tea. If he mixes it in parts proportional to 7 and 5, the value of the mixture is 46 cents a pound. If he mixes it in the ratio of 5 to 7 the value is 50 cents a pound. What is each kind worth per pound?

23. Two trains simultaneously leave the towns P and Q, which are 63 miles apart, and run toward each other. They pass at the end of one hour. The train from P reaches Q 35 minutes before the train from Q reaches P. Find the rate and the time of each train.

24. An express train makes the run between two cities in 5 hours. A way train, whose speed averages 10 miles per hour less, makes the same run in $6\frac{1}{4}$ hours. Find the length of the run and the speed of each train.

25. A man travels 50 miles in an automobile in $3\frac{1}{4}$ hours. If he runs at the rate of 20 miles an hour in the country, and at the rate of 8 miles an hour when within city limits, find how many miles of his journey is in the country.

26. A man starts on a bicycle ride at the rate of 10 miles an hour, intending to ride 20 miles. Owing to an accident he walks part of the way at the rate of 4 miles an hour, and finds himself one hour and a half late. How far does he walk?

27. Two automobiles race, starting together. The first runs at the uniform speed of 30 miles per hour. The second runs at a uniform speed of 35 miles per hour. During the race the faster car is delayed an hour by an accident. Starting again at a uniform speed of 40 miles an hour, it overtakes the first automobile in one hour. How far from the start does this occur?

28. A and B working together can finish a certain piece of work in 12 days. But after working together 6 days, A stops work and B finishes alone in 15 days. What part of the entire work can each do in one day?

29. The length of a room is 8 feet greater than its width. If each be increased by 2 feet, the area of the room will be increased by 60 square feet. What is the actual area of the room?

30. From an 80-gallon cask, filled with wine, a person draws off a certain number of gallons, which he replaces by water. From this mixture he again draws off the same number of gallons as before, and again replaces it by water. It is now found that 16 gallons of the mixture in the cask contains only 9 gallons of wine. How many gallons did he draw off each time?

31. Two automobiles race 336 miles. The winning car wins by 4 hours by going 2 miles an hour faster than the other. What was the winner's time and speed? Explain the negative answer.

32. If 19 pounds of gold and 10 pounds of silver each lose 1 pound when weighed in water, find the amount of each in a mass of gold and silver that weighs 106 pounds in air and 99 pounds in water.

33. How much water must be added to 80 pounds of a 5% salt solution to obtain a 4% solution?

34. A man can walk $2\frac{1}{2}$ miles an hour up hill and $3\frac{1}{2}$ miles an hour down hill. He walks 56 miles in 20 hours on a road no part of which is level. How much of it is up hill?

35. Two launches race to a buoy and return. The first has a start of 10 minutes, steams at the rate of 6 miles per hour to the buoy, and 8 miles per hour on the return. The second steams throughout $7\frac{1}{2}$ miles per hour and is beaten by one minute. How long is the course?

36. A colonel, in attempting to draw up his regiment in the form of a solid square, finds that he has 31 men over, and

that he would require 24 men more in his regiment in order to increase the side of the square by one man. How many men were there in the regiment?

37. There are two numbers such that one is as much greater than 19 as the other is less, and their difference is to their sum as 3 is to 19. Find the numbers.

38. A train makes a run of 120 miles. A second train starts one hour later and travelling 6 miles per hour faster, reaches the end of the same run twenty minutes later than the first train. Find the time of the run of each train.

39. On a certain street railway two sizes of cars are used. What is the seating capacity of each, if fourteen more persons can be seated in three large cars than in four small cars, and two more persons in two large cars than in three small cars?

40. Two men work on a job and each receives 36 dollars. One of them, however, has worked 2 days less than the other and is paid 20 cents more a day. Find his daily wages and the number of days he worked. Explain the negative answer.

41. A number is formed of two digits, whose sum is 6 times their difference. The number itself exceeds 6 times the sum of its digits by 3. Find the number.

42. A and B buy stock, A buying twice as much as B. If A had paid \$1000 more and B \$1000 less, A would have paid three times as much as B. How much money did each invest?

43. Two persons, A and B, run a race to go five times round a certain course. When A has gone three laps, B is 150 yards behind him. A then slackens speed and goes at B's rate, while B quickens his rate and goes at A's first rate. A wins by 30 yards. Find the length of the course, and compare the original speeds of A and B.

XII. SQUARE ROOT

46. *A*

Give the square roots of the following:

1. $36x^2$. **2.** $\frac{4a^4}{25}$. **3.** $a^6b^2c^4$.

4. $4y^6$. **5.** $121m^4n^8$.

6. $x^2 - 4x + 4$.

7. $36a^2 + 9b^2 + 36ab$.

8. $16a^{2m}\,b^{2m+2}$.

9. $\frac{x^2}{16} - \frac{x^5}{10} + \frac{x^8}{25}$.

10. $49m^2 - 126mn + 81n^2$.

B

Extract the square roots of the following:

1. $18a^2 + a^4 + 1 - 8a^3 - 8a$.

2. $4 - 12x - 7x^2 + 24x^3 + 16x^4$.

3. $x^4 + 2x^3y - x^2y^2 - 2xy^3 + y^4$.

4. $1 - 6x + 15x^2 - 18x^3 + 9x^4$.

5. $x^4 + 6x^3 + 11x^2 + 6x + 1$.

6. $9x^4 - 12x^3 + 10x^2 - 4x + 1$.

7. $16x^4 + 16x^3 + 12x^2 + 4x + 1$.

8. $4x^6 + 4x^5 - 3x^4 - 6x^3 - x^2 + 2x + 1$.

9. $a^6 - 2a^5 - a^4 + 3a^2 + 2a + 1$.

10. $a^6 - a^5 - \frac{7a^4}{4} - a^3 + 2a^2 + 2a + 1$.

11. $9x^8 - 12x^7 + 22x^6 - 18x^5 + 19x^4 - 10x^3 + 7x^2 - 2x + 1$.

12. $\frac{x^4}{4} + 4x^2 + \frac{ax^2}{3} + \frac{a^2}{9} - 2x^3 - \frac{4ax}{3}$.

13. $\frac{x^4}{4} + \frac{x^3}{y} + \frac{x^2}{y^2} - xy - 2 + \frac{y^2}{x^2}$.

14. $\frac{4x^2}{9y^2} - \frac{x}{z} - \frac{16x^2}{15yz} + \frac{9y^2}{16z^2} + \frac{6xy}{5z^2} + \frac{16x^2}{25z^2}$.

C

Extract the square roots of the following:

1. 7918596.

2. 1863821584.

3. 291.0436.

4. 25623844.

5. 2250300.01.

6. 56875730.56.

7. 216.15174441.

8. 494210406001.

Extract the square root to the third decimal of the following:

9. 129.

10. .00852.

11. 6.21.

12. 12.

13. $\frac{8}{11}$.

14. $\frac{5}{8}$.

15. $5\frac{1}{6}$.

XIII. EXPONENTS

47.

Express each of the following in radical form:

1. $x^{\frac{1}{2}}$.
2. $3x^{\frac{1}{2}}$.
3. $(2a)^{\frac{1}{2}}$.
4. $xy^{\frac{1}{2}}$.
5. $m^{\frac{2}{3}}n^{\frac{2}{3}}$.
6. $3ab^{\frac{1}{2}}c^{\frac{1}{2}}$.
7. $(x+y)^{\frac{1}{3}}$.
8. $x+3y^{\frac{1}{3}}$.
9. $3a^{\frac{m}{n}}$.
10. $x^{\frac{1}{2}}+y^{\frac{1}{2}}$.

48.

Express each of the following in exponential form:

1. $\sqrt[3]{x}$.
2. $\sqrt[3]{x^2}$.
3. $2\sqrt{a}$.
4. $\sqrt[4]{a^3}$
5. $\sqrt{a^x}$.
6. $\sqrt{3m}$.
7. $\sqrt[n]{a^{2n}}$.
8. $\sqrt[3]{8x^5}$.
9. $\sqrt{a+b}$.
10. $\sqrt{x^2+y^2}$.
11. $3\sqrt[3]{x}+\sqrt[3]{3x}$.
12. $\sqrt[n]{a^n+b^n}$.

49.

Express each of the following with positive exponents:

1. $\dfrac{a^{-1}x}{b}$.
2. $\dfrac{a^{-2}}{b}$.
3. $2a^2b^{-3}$.
4. $2^{-2}x^{-2}$.
5. $-a^3y^{-3}$.
6. $-5a^{-2}b$.
7. $\dfrac{d^{-2}}{m^{-1}n^2}$.
8. $\dfrac{-a^{-2}b^2}{d^3y^{-1}}$.
9. $(a+b)^{-1}$.
10. $a^{-1}+b^{-1}$.
11. $\dfrac{1}{(x-y)^{-2}}$.

12. $\left(\frac{x}{y}\right)^{-2}$.
13. $-\frac{a^{-2}x^3}{m^{-1}}$.
14. $a^{-2}+x^{-1}$.
15. $x^{-2}-xy^{-3}$.
16. x^2-2+x^{-2}.
17. $(-x^2)^{-3}$.
18. $x^{-2}+2x^{-1}y^{-1}+y^{-2}$.
19. $\frac{abc}{(abc)^{-1}}$.
20. $\frac{a^2-b^2-c^2}{3^{-2}ab}$.
21. $\frac{2ab-c}{a^{-2}}$.
22. $\frac{2a+b}{(2a-b)^{-1}}$.
23. $5x^2y^{-2}+2^{-1}xy$.

50.

Find the values of the following:

1. $25^{\frac{1}{2}}$.
2. $64^{\frac{1}{6}}$.
3. $32^{-\frac{1}{5}}$.
4. $(-8)^{-\frac{2}{3}}$.
5. $256^{\frac{3}{4}}$.
6. $216^{-\frac{2}{3}}$.
7. $25^{-\frac{1}{2}}$.
8. $(-8)^{\frac{5}{3}}$.
9. $81^{\frac{3}{4}}$.
10. $49^{-\frac{1}{2}}$.
11. $(-512)^{\frac{2}{9}}$.
12. $-32^{\frac{3}{5}}$.
13. $125^{-\frac{2}{3}}$.
14. $(-1728)^{-\frac{1}{3}}$.
15. $25^{-\frac{3}{2}}$.
16. $(\frac{25}{16})^{-\frac{1}{2}}$.
17. $(\frac{1}{8\frac{1}{2}})^{-\frac{2}{5}}$.
18. $(1\frac{2}{7}\frac{5}{7})^{\frac{2}{3}}$.
19. $(\frac{1}{4})^{\frac{3}{2}}$.
20. $(\frac{16}{49})^{-\frac{3}{2}}$.
21. $(5\frac{1}{16})^{\frac{3}{4}}$.
22. $(15\frac{5}{8})^{-\frac{2}{3}}$.
23. $(\frac{25}{49})^{-\frac{3}{2}}$.
24. $(\frac{8}{27}\frac{2}{48})^{-\frac{1}{3}}$.
25. $(.001)^{\frac{1}{3}}$.
26. $(.125)^{-\frac{1}{3}}$.
27. $(1.44)^{-\frac{1}{2}}$.
28. $-16^{\frac{1}{2}}$.
29. $(-8)^{-\frac{2}{3}}$.

51.

Perform the indicated operations, expressing in simplest form:

1. $x^4 \cdot x^{-2}$.
2. $x^{-4} \cdot x^2$.
3. $a^{\frac{1}{2}} \cdot a^{\frac{1}{3}}$.

4. $m^{-\frac{1}{2}} \cdot m^{\frac{1}{4}}$. 5. $2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}$. 6. $3^{\frac{1}{2}} \cdot 3^{-\frac{1}{3}}$.

7. $a^{-3} \cdot a^{-4}$. 8. $9^{-\frac{1}{2}} \cdot 27^{\frac{1}{3}}$. 9. $8^{\frac{1}{3}} \cdot 2^{-2}$.

10. $-27^{\frac{2}{3}} \cdot 3^{-5}$. 11. $b^{m-3} \cdot b^{2}$.

12. $2a^{-3} \cdot 3a^{-2}$. 13. $2a^{-2} \cdot 2a^{2}$.

14. $81^{-\frac{3}{4}} \cdot 3^{3}$. 15. $y^{2n} \cdot y^{-3n} y^{-n}$.

16. $x^{-5} \div x^{2}$. 17. $a^{-2} \div a^{-1}$.

18. $c^{x} \div c$. 19. $x^{a-b} \div x^{b-a}$

20. $(x + y)^{a-b} \div (x + y)^{b-a}$.

21. $(\frac{1}{2})^{-3} \div (\frac{1}{2})$.

22. $(\frac{1}{4})^{-\frac{3}{2}} \div (\frac{1}{8})^{\frac{2}{3}}$.

23. $8x^{\frac{1}{3}} \div (-2x^{-1})$.

24. $-2d^{-3} \div (-4d^{-5})$.

52.

Simplify the following:

1. $(2^{2})^{-2}$. 2. $(3^{-2})^{-2}$. 3. $(7^{\frac{1}{3}})^{6}$.

4. $-(8^{\frac{1}{6}})^{2}$. 5. $(9^{-\frac{1}{3}})^{\frac{3}{2}}$. 6. $(-8^{-\frac{2}{3}})^{2}$.

7. $2(x^{-1}y^{\frac{1}{3}})^{3}$. 8. $(2a \div 3a^{-1})^{-2}$.

9. $(5a^{n-1}b^{2})^{2}$. 10. $(y^{2x})^{x}$.

11. $(b^{3x})^{-\frac{1}{6}}$. 12. $(2a^{\frac{1}{2}}x^{-1}y)^{-2}$.

13. $(x^{-\frac{1}{m}}x^{\frac{1}{n}})^{mn}$.

14. $(a^{2b} \div a^{-b})^{\frac{1}{b}}$. 15. $(3a^{-2})^{-2x}$.

16. $2^{-3} \cdot (\frac{3}{4})^{-2}$. 17. $[(4a)^{-2}]^{2}$.

18. $(9^{-3}m^{-4}n^{-6})^{-\frac{1}{3}}$.

53.

Express in simplest form, using the rules governing exponents:

1. $\dfrac{a\sqrt{a}}{b\sqrt{b}}$.
2. $a^{\frac{1}{2}}\sqrt{x}\,a^{\frac{1}{3}}\sqrt{x^{-1}} \div a\sqrt{x^3}$.
3. $m\sqrt[4]{m^2n^4w^6}$.
4. $[\sqrt[n]{a^{n-1}b^n}]^{2n}$.
5. $a^2\sqrt{x}\cdot a^{\frac{1}{2}}x^{-1}$.
6. $\sqrt[3]{a\sqrt{b}}$.
7. $\dfrac{a^{\frac{1}{3}}\sqrt{a^2\sqrt{x}}}{a^2x}$.
8. $\dfrac{2a^{-\frac{1}{2}}\sqrt{m^3}}{3m^{-\frac{1}{2}}\sqrt{a^3}}$.
9. $\dfrac{a^2c^{\frac{2}{3}}\sqrt{a^{-1}}}{\sqrt{ac\sqrt{ac}}}$.
10. $\sqrt[3]{\sqrt{\sqrt[3]{m^{ax}}}}$.
11. $\dfrac{x^{\frac{m}{3}}\sqrt[m]{a^{-3}}}{a^{\frac{1}{m}}\sqrt{x^{-m}}}$.
12. $\dfrac{a^{n(n-1)}}{a^n(a^{\frac{2}{3}})^0}$.
13. $\dfrac{\sqrt[3]{x^{-1}\sqrt{y^3}}}{\sqrt{y^{-1}\sqrt[3]{x^2}}}$.
14. $\left(\dfrac{25\sqrt{x}}{9\sqrt[3]{x^{-2}}}\right)^{-\frac{3}{2}}$.

54.

Multiply by inspection:

1. $(x^{-1} + y^{-1})(x^{-1} - y^{-1})$.
2. $(a^{-1} - b)(a^{-1} + b)$.
3. $(a^{-1} - b^{-1})^2$.
4. $(x^{-2} + y^{-2})^2$.
5. $(x^{\frac{1}{3}} - 4)(x^{\frac{1}{3}} - 3)$.
6. $(x^{-\frac{1}{2}} + x^{\frac{1}{2}})^2$.
7. $(x^{-\frac{1}{2}} + 6)(x^{-\frac{1}{2}} - 5)$.

8. $(m^{-\frac{1}{3}} - m^{-1})(2m^{-\frac{1}{3}} - m^{-1})$.

9. $(a^{-2} - a^{-1} - 1)^2$.

10. $(x^{-1} - y)(x^{-1} + y)(x^{-2} + y^2)$.

11. $(a^{-\frac{m}{2}} + 3a^{\frac{m}{2}})^2$.

Multiply:

12. $x^{-2} + 2x^{-1} + 3$ by $x^{-2} + x^{-1} - 2$.

13. $x^{\frac{1}{2}} + x + x^{\frac{3}{2}} - 1$ by $x^{\frac{1}{2}} - x - x^{\frac{3}{2}} + 1$.

14. $\sqrt{a^3} - 2a + \sqrt{a} + \dfrac{1}{\sqrt{a}}$ by

$$\frac{1}{\sqrt{a^3}} - \frac{1}{a} + \frac{1}{\sqrt{a}} + \sqrt{a}.$$

15. $m^{\frac{2}{3}} - 2a^{\frac{1}{2}}m^{\frac{1}{3}} + 4a^{\frac{1}{2}}$ by $m^{\frac{2}{3}} + 2a^{\frac{1}{2}}m^{\frac{1}{3}} + 4a^{\frac{1}{2}}$.

16. $x^{\frac{2}{3}} - 4 + 4x^{-\frac{2}{3}}$ by $x^{\frac{2}{3}} + 4 + 4x^{-\frac{2}{3}}$.

17. $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.

18. $a^{\frac{5}{2}} + a^{\frac{3}{2}} + a^{-\frac{3}{2}} + a^{-\frac{1}{2}} + a^{\frac{1}{2}}$ by $a^{\frac{1}{2}} - a^{-\frac{1}{2}}$.

19. $2\sqrt{x} + 3x^{\frac{1}{6}}\sqrt[3]{y} + \dfrac{4\sqrt[3]{y^2}}{\sqrt[6]{x}}$ by

$$\frac{3}{\sqrt{x}} - \frac{3}{\sqrt[6]{xy^2}} + \frac{2\sqrt[6]{x}}{\sqrt[3]{y^2}}.$$

Divide, obtaining the quotient by inspection wherever possible:

20. $x^{-2} - y^{-2}$ by $x^{-1} - y^{-1}$.

21. $x^{-1} - x^{-\frac{1}{2}} - 6$ by $x^{-\frac{1}{2}} - 3$.

22. $a^{-2} - 4a^{-1}b^{-1} + 4b^{-2}$ by $a^{-1} - 2b^{-1}$.

23. $x^{-3} - 8y^6$ by $x^{-1} - 2y^2$.

24. $a + b$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}}$.

25. $a - a^{\frac{1}{4}} b^{\frac{3}{4}}$ by $a^{\frac{1}{4}} - b^{\frac{1}{2}}$.

26. $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}} y - y^{\frac{3}{2}}$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.

27. $x^2y^{-3} - xy^{-1} + x^{-1}y - xy^{-2} + x^{-2}y - x^{-3}y^3$ by $x^{-1}y^{-2} - x^{-2}y^{-1}$.

28. $x - 5\sqrt[3]{x^2} - 40 - 46\sqrt[3]{x}$ by $4 + \sqrt[3]{x}$.

29. $\frac{9}{a} - \frac{3\sqrt{x}}{\sqrt[4]{a^3}} + \frac{10x}{\sqrt{a}} + \frac{\sqrt{x^3}}{\sqrt[4]{a}} - x^2$ by $\frac{3}{\sqrt[4]{a}} + \sqrt{x}$.

Find the square roots of:

30. $m^{-1} - 22m^{-\frac{1}{2}} n^{-\frac{1}{4}} + 121n^{-\frac{1}{2}}$.

31. $x^{\frac{5}{3}} - 4x^{\frac{4}{3}} + 2x + 4x^{\frac{2}{3}} + x^{\frac{1}{3}}$.

32. $\sqrt[3]{a^2} - 4\sqrt[6]{a^5} + 4a + 2a\sqrt[6]{a} - 4a\sqrt[3]{a} + a\sqrt[3]{a^2}$.

33. $x^{\frac{8}{5}} - 2a^{-\frac{8}{5}} x^{\frac{11}{5}} + 2a^{\frac{4}{5}} x^{\frac{4}{5}} + a^{-\frac{6}{5}} x^{\frac{14}{5}} - 2a^{\frac{1}{5}} x^{\frac{7}{5}} + a^{\frac{8}{5}}$

34. $9x^5 - \frac{24\sqrt{x^7}}{\sqrt[3]{y}} + \frac{4x^2}{\sqrt[3]{y^2}} + 16\sqrt{\frac{x}{y^2}} + \frac{4}{xy\sqrt[3]{y}}$.

35. $1 + x + 2x^2$ to 4 terms.

36. $x^2 - 1$ to 4 terms.

55.

Solve the following equations:

1. $x^{\frac{1}{4}} = 3$.

2. $x^{\frac{2}{3}} = 16$.

3. $x^{\frac{3}{5}} = -8$.

4. $x^{-\frac{3}{4}} = -27$.

5. $x^{-\frac{2}{3}} = 4$.

6. $x^{-1} = 2$.

7. $x^{\frac{5}{2}} = -32$.

8. $x^{-\frac{1}{2}} = 5$.

9. $x^{\frac{3}{2}} = 8$.

10. $2^x = 16$.

11. $4^x = 32$.

12. $9^x = \frac{1}{27}$.

13. $(\frac{1}{3})^{-x} = 27$.

14. $(x + 2)^3 = 125$.

15. $(x - 3)^{\frac{3}{2}} = 8$.

16. $(x^2 - 1)^{-2} = \frac{1}{9}$.

17. $(x^{-\frac{2}{3}} - 3)^2 = 1$.

56.

Simplify expressing with positive exponents:

1. $\dfrac{a^{-1}+b^{-1}}{a}$.

2. $\dfrac{2a^{-2}+b}{a^{-2}b^2}$.

3. $(16a^{-4} \div 9a^{-2})^{-\frac{1}{2}}$.

4. $(x^{\frac{1}{2}}y^{\frac{2}{3}})^{\frac{1}{3}} \div (x^{-\frac{1}{3}}y)^{\frac{1}{2}}$.

5. $\left(\dfrac{x^{n+1}}{x}\right)^n \div \left(\dfrac{x}{x^{1-n}}\right)^{n-1}$.

6. $\left[\sqrt[4]{\left(-\dfrac{27}{64}\right)^{-\frac{2}{3}}}\right]^{-\frac{2}{5}}$.

7. $(a^9b^{12})^{\frac{1}{3}} + [(a^2b^8)^{\frac{1}{4}}]^2 + [(a^3b^7)^{\frac{1}{2}}]^{\frac{1}{11}}$.

8. $\{a^{\frac{1}{2}}b^{-\frac{1}{4}}[a^{\frac{1}{3}}b^{\frac{1}{2}}(b^{\frac{2}{3}})^{\frac{1}{2}}]^{\frac{1}{2}}\}^3$.

9. $\left(\dfrac{x^{m+n}}{x^n}\right)^m \div \left(\dfrac{x^n}{x^{m+n}}\right)^{n-m}$.

10. $(2^{n+4} - 2 \times 2^n) \div (4 \times 2^{n+2})$.

11. $\left[\left(\dfrac{ay}{x^3}\right)^{\frac{1}{2}} \times \left(\dfrac{bx^2}{y^4}\right)^{\frac{1}{3}} \times \left(\dfrac{y^3}{a^{\frac{5}{2}}b^{\frac{5}{2}}}\right)^{\frac{1}{5}}\right]^{\frac{6}{7}}$.

12. $\dfrac{3^{x-1} \times (3^{-1})^{x+1}}{(3^{-x})^{-1} \times (3^{x-2})^{-1}}$.

13. $\left\{\dfrac{a^{-\frac{1}{2}}b^{-\frac{1}{3}}}{a^{-\frac{3}{4}}b^{-\frac{5}{6}}} \div (a^{-3}b^{-5})^{\frac{1}{4}}\right\}^{\frac{2}{7}}$.

14. $\left\{\dfrac{x^{\frac{5}{2}}y^{\frac{4}{3}}}{z^{-\frac{5}{4}}} \times \dfrac{z^4}{x^{-3}y^{-\frac{5}{3}}} \div \dfrac{y^{-2}z^{\frac{1}{4}}}{x^{-\frac{1}{2}}}\right\}^{\frac{1}{5}}$.

15. $[(a^{\frac{1}{2}})^{\frac{2}{3}-\frac{1}{6}} - \frac{3}{4}(a^{\frac{5}{2}}b\{a^{-3}b^{-2}\}^{\frac{1}{2}})^{\frac{1}{4}}]^4$.

16. $\left\{\dfrac{(x^{-1})^{\frac{1}{4}}}{y^{\frac{4}{3}}} \times \left(\dfrac{x^{\frac{1}{3}}}{y^{-\frac{1}{2}}}\right)^2 \div \dfrac{y^{-\frac{2}{3}}}{x^{-\frac{1}{2}}}\right\}^{12}$

17. $\left\{\left(\dfrac{a^{-\frac{1}{2}}b^2}{b^{\frac{1}{2}}a}\right)^{\frac{1}{3}} \times \left(\dfrac{a^{-1}b}{(ab^2)^{\frac{1}{2}}}\right)^{\frac{1}{3}}\right\}^{-2}.$

18. $\dfrac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{c^{\frac{1}{6}}} \div \left\{\dfrac{c^{-\frac{1}{2}}}{a^{-\frac{1}{3}}b^{-\frac{1}{3}}} \times \dfrac{a^{-\frac{5}{6}}c^{-\frac{2}{3}}}{b^{\frac{5}{6}}}\right\}.$

19. $\dfrac{x^2y^{-3} - x^{-1}}{xy^{-2} + y^{-1} + x^{-1}}.$ 20. $\dfrac{1 + \left(\dfrac{a+x}{a-x}\right)^{-2}}{1 - \left(\dfrac{a+x}{a-x}\right)^{-2}}.$

21. $\dfrac{\left(\dfrac{1}{x^{-3}} + \dfrac{1}{y^{-3}}\right)^{-1}(x^2 - xy + y^2)}{\left(\dfrac{1}{x^{-3}} - \dfrac{1}{y^{-3}}\right)^{-1}(x^2 + xy + y^2)}.$

22. $\dfrac{1 + \left(\dfrac{a+x}{a-x}\right)^{-1}}{1 - \left(\dfrac{a+x}{a-x}\right)^{-1}} \div \dfrac{1 + \left(\dfrac{a^2+x^2}{a^2-x^2}\right)^{-1}}{1 - \dfrac{a^2-x^2}{a^2+x^2}}.$

23. $\dfrac{a^{-2} - b^{-2}}{a^{-3} - b^{-3}}.$ 24. $\dfrac{a^{-2}b^{-2}}{a^{-2} + b^{-2}} \times \dfrac{a^2 + b^2}{a^{-1} - b^{-1}}$

25. $\dfrac{3a^{n+1}}{a^{n-1}x} \times \dfrac{x^{-5}yz^m}{a^2y^{-1}z^{m-1}} \times \dfrac{9z^2}{y^2z^3}.$

26. $\dfrac{x^{-1}y^{-1}z^{-n}}{ax^{-2}y^{-3}} \times \dfrac{a^nb^{n-1}c^{-n}}{x^{-3}yz} \times \dfrac{a^{-n}bc^{n-1}}{x^2yz^{1-n}}.$

27. $\dfrac{x^3 + x^{-3} - 5(x + x^{-1})}{x + x^{-1}}.$

28. $\frac{1}{4}(xa^{-1} - ax^{-1})\left(\dfrac{a^{-1} - x^{-1}}{a^{-1} + x^{-1}} - \dfrac{a^{-1} + x^{-1}}{a^{-1} - x^{-1}}\right).$

29. $\dfrac{(a^2b^{-2} + b^2a^{-2} + 1)(a^2b^{-2} + b^2a^{-2} - 1)}{a^4b^{-4} + b^4a^{-4} + 1}.$

30. $8^{-\frac{2}{3}} + 9^{\frac{3}{2}} - 2^{-2} + 1^{-\frac{2}{5}} - 7^{0}$.

31. $\dfrac{2^{n+4} - 2 \cdot 2^{n}}{2 \cdot 2^{n+3}}$.

32. $\dfrac{3^{n} \cdot (3^{n-1})^{n}}{3^{n+1} \cdot 3^{n-1} \cdot 9^{-n}}$.

33. $2^{\frac{2}{3}} \cdot 6^{\frac{1}{3}} - 9^{-\frac{2}{3}} \cdot 3^{\frac{5}{3}} - \dfrac{576^{\frac{1}{6}}}{2}$.

34. $\sqrt[3]{2 \cdot 4^{-5}} - \dfrac{\sqrt[7]{2}}{2^{-\frac{6}{7}}} + \dfrac{1}{8^{-\frac{4}{3}}} - 128^{-\frac{3}{7}}$.

35. $\left\{ \sqrt[4]{\dfrac{n^{-\frac{1}{3}} a^{3}}{81 a^{-1} \sqrt{x}}} \cdot \sqrt[3]{\dfrac{8x^{\frac{1}{2}} \sqrt{a^{3} n}}{x^{\frac{1}{4}} a^{\frac{3}{4}} \sqrt[8]{n^{5}}}} \right\}^{-1}$.

36. $\left[x^{-6} - \left(\dfrac{1}{y^{-1}} \right)^{-2} \right] \div \left[x^{-3} + \left(\dfrac{1}{y^{-1}} \right)^{-1} \right]$.

XIV. RADICALS

Reduction to Simplest Form

CASE I

57. When the radicand contains a factor which is a perfect power of the same degree as the radical.

Reduce to simplest radical form:

1. $5\sqrt{20}$.
2. $\frac{1}{2}\sqrt[3]{32}$.
3. $2\sqrt{80}$.
4. $2a\sqrt{28b^2}$.
5. $\sqrt[3]{432}$.
6. $\sqrt{288}$.
7. $\sqrt{1350}$.
8. $\sqrt{6125}$.
9. $\sqrt[3]{-1029}$.
10. $\sqrt{2880}$.
11. $\sqrt{127008}$.
12. $\sqrt{a^3 + 4a^2b + 4ab^2}$.
13. $\sqrt[3]{216 - 648x^2}$.
14. $\sqrt{8x^3 - 24x^2y + 18xy^2}$.
15. $\sqrt{3x^2y + 12xy + 12y}$.
16. $\sqrt{x^3 - x^2y - xy^2 + y^3}$.
17. $\sqrt{x^4 - x^3y - xy^3 + y^4}$.
18. $3\sqrt[3]{3x^4y - 9x^3y^2 + 9x^2y^3 - 3xy^4}$.
19. $\sqrt{x^4 - x^3y - x^2y^2 + xy^3}$.
20. $\sqrt[3]{2a^4b - 6a^3b^2 + 6a^2b^3 - 2ab^4}$.
21. $\sqrt{x^5 - 2x^3y^2 + xy^4 + x^4y - 2x^2y^3 + y^5}$.
22. $\sqrt[3]{27x^4y + 81x^2y^3 - 81x^3y^2 - 27xy^4}$.
23. $2\sqrt[5]{32a^5x - 64a^5y}$.
24. $\frac{2}{3}\sqrt{3ax^2 - 30ax + 75a}$.
25. $\sqrt[n]{a^{n+1}}$.
26. $2x\sqrt[n]{x^{2n+1}}$.
27. $\sqrt[3]{x^6y^{6+x}}$.

Case II

58. When the radicand is a fraction.

Express in simplest form:

1. $2\sqrt{\frac{7}{9}}$.
2. $3\sqrt{\frac{1}{2}}$.
3. $5\sqrt{\frac{2}{3}}$.
4. $\sqrt{\frac{3}{4}}$.
5. $\sqrt[3]{\frac{2}{3}}$.
6. $5\sqrt[3]{\frac{1}{2}}$.
7. $\sqrt{\frac{a-b}{a+b}}$.
8. $\frac{5ab}{2}\sqrt{\frac{20}{25a^2b}}$.
9. $\frac{4a}{x}\sqrt[3]{\frac{27x^4}{4a^2}}$.
10. $\sqrt[n]{\frac{a^{n+1}}{b}}$.
11. $a\sqrt{\frac{1}{a}+\frac{1}{b}}$.
12. $\sqrt[2a]{\frac{y}{x^2}}$.
13. $(x-y)\sqrt{\frac{x+y}{(x-y)^3}}$.
14. $\frac{ab}{x^2-y^2}\sqrt{\frac{(x-y)^3}{ab}}$.
15. $\frac{x^2-y^2}{3x-2y}\sqrt{\frac{27x^2-36xy+12y^2}{x-y}}$.
16. $2\sqrt{\frac{9(a-b)}{2(a+b)}}$.
17. $\sqrt[n]{\frac{x^{3n+2}y^5}{y^{2n}}}$.
18. $\frac{2y^2}{x^n}\sqrt{\frac{x^{2n+1}}{y}}$.
19. $\frac{b^2}{a^2}\sqrt[n]{\frac{a^{n+1}}{b^{n+1}}}$.

Case III

59. When the radicand is a perfect power whose exponent is a factor of the index of the radical.

Reduce the following to their simplest forms:

1. $\sqrt[4]{(a+b)^2}$.
2. $\sqrt[6]{8}$.
3. $\sqrt[10]{(a+b)^5x^{10}y^5}$.
4. $\sqrt[12]{64a^6b^{18}}$.
5. $\sqrt[2m]{4a^{2m}b^{4m}c^2}$.
6. $\sqrt[3m+3]{8x^3y^{3m}}$.
7. $\sqrt[4]{16a^{4m}b^{-8m}}$.
8. $\sqrt[6]{216a^{-3}}$.
9. $\sqrt[15]{343a^3b^{-6}}$.
10. $\sqrt[4]{x-2x^{\frac{1}{2}}y^{\frac{1}{2}}+y}$.
11. $\sqrt[n]{a^{2n}b^{2n}c^{mn}}$.
12. $\sqrt[6]{64a^8b^{10}}$.

13. $\sqrt[6]{36x^2 + 120xy + 100y^2}$.

14. $\sqrt{(x - y)^6(x + y)^3}$.

15. $\sqrt[6]{8a^3x^3 - 48a^2x^3 + 96ax^3 - 64x^3}$.

16. $\sqrt[12]{4a^2 - 24ax + 36x^2}$.

Reduction to Different Forms

Case I

60. Express as entire surds:

1. $5a^2\sqrt{a}$.

2. $-2a\sqrt{3x}$.

3. $(x - y)\sqrt{\frac{1}{x - y}}$.

4. $\frac{1}{2}\sqrt{8}$.

5. $\frac{x - y}{x + y}\sqrt{\frac{x + y}{x - y}}$.

6. $\frac{1}{x - y}\sqrt[3]{x^3 - y^3}$.

7. $2ax\sqrt[4]{\frac{1}{4a^2x^2}}$.

8. $\frac{1}{x + 5}\sqrt{x^2 + 8x + 15}$.

9. $(x + y)\sqrt{\frac{x - y}{x^3 + y^3}}$.

Case II

61. Change the following to radicals of the required index:

1. $2ab$ to the 4th.

2. $3a^2b^2$ to the 3d.

3. $\sqrt{a + b}$ to the 10th.

4. $\sqrt[3]{12x^2}$ to the 6th.

5. $\sqrt[4]{4(a - x)}$ to the 8th.

6. $2(a + x)$ to the 3d.

7. $(a + b)^2$ to the mth.

8. $\sqrt[3]{(a + b)^2(a - b)^2}$ to the 6th.

9. $\sqrt[n]{(a + b)^2 + c}$ to the mth.

62. Change to equivalent radicals of the same index:

1. $\sqrt[3]{3}, \sqrt{2}.$
2. $\sqrt{3}, \sqrt[4]{5}.$
3. $\sqrt{2}, \sqrt[3]{3}, \sqrt[4]{3}.$
4. $\sqrt[m]{a}, \sqrt[n]{b}.$
5. $\sqrt[3]{a^2}, \sqrt[4]{b^3}, \sqrt[5]{c^4}.$
6. $\sqrt[5]{\frac{2}{3}}, \sqrt[15]{\frac{1}{2}}.$
7. $\sqrt{a}, \sqrt[3]{b}, \sqrt{b}, \sqrt[3]{c^2}, \sqrt[m]{c^{\frac{3m}{2}}}.$
8. $(a+b)^{\frac{1}{m}}, (b+c)^{\frac{m}{n}}, (c+d)^{\frac{m}{2n}}.$

Which is greater:

9. $\sqrt{3}$ or $\sqrt[3]{4}$?
10. $2\sqrt{5}$ or $3\sqrt[3]{11}$?
11. $\sqrt{\frac{1}{2}}$ or $\sqrt[3]{\frac{1}{2}}$?

Arrange in order of magnitude:

12. $\sqrt{3}, \sqrt[3]{4}, \sqrt[4]{7}.$
13. $\sqrt[4]{5}, \sqrt[3]{6}, \sqrt{3}.$
14. $3\sqrt[3]{5}, \sqrt[6]{500}, 2\sqrt{8}.$
15. $\sqrt{2}, \sqrt[4]{6}, \sqrt[3]{2\frac{1}{2}}.$
16. $12\sqrt[4]{3}, 5\sqrt{6}, 4\sqrt{5}.$
17. $2\sqrt{3}, 3\sqrt[3]{2}, 2\sqrt[3]{6}.$
18. $4\sqrt[3]{2}, 3\sqrt[3]{7}, 4\sqrt{2}.$

63. Addition and Subtraction of Radicals.

Combine the following:

1. $\sqrt{8} + \sqrt{32}.$
2. $2\sqrt{50} - \sqrt{18}.$
3. $\sqrt{a^2b} + \sqrt{a^4b}.$
4. $4\sqrt[3]{16} - 2\sqrt[3]{54}.$
5. $4\sqrt{72} + 3\sqrt{32} + 4\sqrt{8} + 2\sqrt{18}.$
6. $3\sqrt{80} + \sqrt{45} + \sqrt{245} + 2\sqrt{45}.$
7. $2\sqrt[3]{54} + \sqrt[3]{250} + 4\sqrt[3]{16} + 3\sqrt[3]{54}.$

8. $\frac{1}{2}\sqrt{75} - \frac{1}{4}\sqrt{27} - \frac{1}{6}\sqrt{3} + \frac{1}{3}\sqrt{12}$.

9. $\frac{1}{2}\sqrt{\frac{3}{4}} + \frac{1}{2}\sqrt{\frac{1}{3}} - \frac{1}{2}\sqrt{\frac{8}{25}}$.

10. $2\frac{3}{8}\sqrt{\frac{5}{4}} + 5\sqrt{\frac{5}{16}} + 3\sqrt{4\frac{5}{16}}$.

11. $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{8}} + \frac{1}{4}\sqrt{18}$.

12. $\sqrt[4]{\frac{1}{3}} + \frac{1}{3}\sqrt[4]{\frac{1}{48}} + \sqrt[4]{\frac{32}{486}}$.

13. $5\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{8}} - \sqrt{\frac{4}{18}}$.

14. $2\sqrt{m^2n^2x} + 3m\sqrt{x^3} + m\sqrt{25x^5}$.

15. $2\sqrt{ax^2 - 4a^2x + 4a^3} + \sqrt{16a^3}$.

16. $(8a^3 - 24a^2 + 18a)^{\frac{1}{2}} - (2a^3 - 12a^2 + 18a)^{\frac{1}{2}}$.

17. $\sqrt{(a+b)(a-b)(a+b)^{-2}} + \sqrt{\frac{a+b}{a-b}}$.

18. $\frac{x-y}{x+y}\sqrt{\frac{x+y}{x-y}} + \sqrt{9(x^2-y^2)(x+y)^{-2}}$.

19. $\sqrt[n]{\frac{xy}{(x+y)^{-n}}} + \sqrt[n]{\frac{(x-y)^n}{(xy)^{-1}}}$.

64. Multiplication of Radicals.

Multiply the following, expressing results in simplest form:

1. $\sqrt{2} \times \sqrt{6}$.
2. $\sqrt[3]{3} \times \sqrt[3]{9}$.
3. $2\sqrt{3} \times \sqrt{6}$.
4. $5\sqrt{12} \times 3\sqrt{8}$.
5. $2\sqrt[3]{32} \times 5\sqrt[3]{12}$.
6. $\frac{1}{2}\sqrt{20} \times \frac{1}{3}\sqrt{18}$.
7. $\frac{2}{3}\sqrt[3]{54} \times 3\sqrt[3]{64}$.
8. $a^2b\sqrt{ab^2} \times ab^2\sqrt{a^2b}$.
9. $\frac{1}{2}\sqrt[4]{8} \times \frac{1}{3}\sqrt[4]{160}$.
10. $(2\sqrt{8} + 2\sqrt{18})(\sqrt{75} + 4\sqrt{24})$.
11. $(9\sqrt{162} + 3\sqrt{50} + 5\sqrt{2} - 3\sqrt{32})\sqrt{2}$.
12. $5\sqrt{12} \times 3\sqrt{8} \times 6\sqrt{2} \times \sqrt{3}$.

13. $\sqrt[5]{16} \times \sqrt[5]{2} \times \sqrt[5]{96} \times 2\sqrt[5]{12}$.

14. $(3\sqrt{45} - \sqrt{20} + 7\sqrt{5})(4\sqrt{63} + 5\sqrt{7} - 8\sqrt{28})$.

15. $(2\sqrt{8} + \sqrt{\frac{9}{2}} + 6\sqrt{18})(\sqrt{\frac{25}{8}} + \sqrt{12} - \sqrt{\frac{4}{3}})$.

16. $(\sqrt{25a} + 3\sqrt{ab^2} + 2\sqrt{9a})(11\sqrt{a} + b\sqrt{9a} - 3\sqrt{4a})$

17. $\sqrt{50}(8 - \sqrt{39})^{\frac{1}{2}}(8 + \sqrt{39})^{\frac{1}{2}}$.

18. $(3\sqrt{80} - 4\sqrt{20} - \sqrt{45})^2$.

19. $(\sqrt{x+1} + 2)^2$.

20. $(\sqrt{x+1} + \sqrt{x-1})^2$.

21. $(5\sqrt{x} - 2\sqrt{x-1})^2$.

22. $(\frac{1}{2}\sqrt{x} - 12)^2$.

23. $\sqrt{5 - \sqrt{21}} \times \sqrt{5 + \sqrt{21}}$.

24. $\sqrt{6 - 3\sqrt{2}} \times \sqrt{6 + 3\sqrt{2}}$.

25. $(2\sqrt{5} - 3)(4\sqrt{5} + 7)$.

26. $(3\sqrt{7} + 2\sqrt{3})(3\sqrt{7} - 5\sqrt{3})$

27. $(2\sqrt{2} - \sqrt{3} + \sqrt{5})(3\sqrt{2} + 2\sqrt{3} + 2\sqrt{5})$.

28. $(6\sqrt{6} - 2\sqrt{2} - 8\sqrt{3})(3\sqrt{6} - \sqrt{2} + 4\sqrt{3})$.

29. $\sqrt{15} \times \sqrt[3]{18}$.

30. $\sqrt{3} \times \sqrt[3]{4} \times \sqrt[4]{12}$.

31. $\sqrt{a^2b} \times \sqrt[3]{ab} \times \sqrt[5]{ab^2}$.

32. $\sqrt[3]{a+b} \times \sqrt{a-b} \times \sqrt[4]{a+b}$.

33. $2\sqrt{5} \times 5\sqrt[3]{10} \times 3\sqrt[6]{3}$.

34. $\sqrt[3]{25} \times \sqrt[3]{50} \times \sqrt[4]{15}$.

35. $\sqrt[4]{25} \times \sqrt[6]{27} \times \sqrt[8]{16}$.

36. $\sqrt[5]{64} \times \sqrt[10]{9} \times 3\sqrt[15]{27}$.

37. $\sqrt[5]{a^2x} \times \sqrt[6]{ax} \times \sqrt[6]{a^2x^2}$.

38. $\sqrt[3]{a^3 - x^3} \times \sqrt{a^2 - x^2} \times \sqrt{a - x}$.

39. $\sqrt{a} \times \sqrt[3]{a} \times \sqrt[4]{a} \times \sqrt[5]{a}$.

40. $\sqrt[3]{\frac{3}{4}} \times \sqrt{\frac{1}{2}}$.

41. $\sqrt[5]{\frac{27}{128}} \times \sqrt[6]{\frac{16}{9}}$.

42. $\sqrt[9]{\frac{17}{16}} \times \sqrt[4]{\frac{2}{3}}$.

43. $\sqrt[5]{\frac{25}{16}} \times \sqrt{\frac{3}{5}}$.

44. $\sqrt[4]{\frac{16}{81}} \times \sqrt[3]{\frac{9}{4}}$.

65. Division of Radicals.

Divide the following, expressing results in simplest form:

1. $\sqrt{8} \div \sqrt{2}$.

2. $\sqrt{15} \div \sqrt{3}$.

3. $2\sqrt{3} \div 3\sqrt{2}$.

4. $8\sqrt{243} \div 2\sqrt{81}$.

5. $\frac{1}{2}\sqrt{225} \div \sqrt{\frac{5}{4}}$.

6. $(3\sqrt{8} + 5\sqrt{2}) \div 3\sqrt{2}$.

7. $(72\sqrt{20} + 32\sqrt{45}) \div 2\sqrt{5}$.

8. $\sqrt{\frac{8}{11}} \div \sqrt{\frac{3}{2}}$.

9. $\sqrt{\frac{2}{3}} \div \sqrt{\frac{25}{8}}$.

10. $3\sqrt{a^2bc} \div \sqrt{4a^5bcx}$.

11. $\sqrt{\frac{1}{2}} \div \sqrt[3]{\frac{2}{3}}$.

12. $\sqrt{12} \div \sqrt[4]{10}$.

13. $(2\sqrt{8} \times 3\sqrt{3}) \div (\sqrt{45} \times 2\sqrt{32})$.

14. $\sqrt[3]{8\frac{1}{8}} \div \sqrt{3\frac{1}{2}}$.

15. $(\sqrt{12} + \sqrt{300}) \div (\sqrt{8} + \sqrt{32})$.

16. $\sqrt{a^3 - b^3} \div \sqrt{(a - b)^3}$.

17. $16 \div \sqrt[3]{768}$.

18. $\sqrt[5]{1600} \div 2$.

19. $\sqrt[3]{64} \div \sqrt{8}$.

20. $\sqrt[3]{128} \div \sqrt{\frac{1}{2}}$.

21. $\dfrac{3\sqrt{48}}{5\sqrt{112}} \div \dfrac{6\sqrt{84}}{\sqrt{392}}$.

22. $\dfrac{3\sqrt{5}}{2\sqrt{98}} \div \dfrac{15}{7\sqrt{15}}$.

23. $\sqrt[4]{\frac{9}{4}} \div \sqrt[9]{\frac{27}{64}}$.

24. $\sqrt[3]{\frac{2}{3}} \div \sqrt[10]{\frac{32}{243}}$.

25. $\sqrt[5]{\frac{16}{27}} \div \sqrt[3]{\frac{4}{9}}$.

66. Involution of Radicals.

Expand the following:

1. $(\sqrt[4]{12})^3$.
2. $(\sqrt{6})^4$.
3. $(\sqrt[3]{15})^4$.
4. $(\sqrt[12]{64})^3$.
5. $(5\sqrt{3a^2bc})^3$.
6. $(\sqrt[3]{a^2-b^2})^{12}$.
7. $(\sqrt{(a+b)^3})^4$.
8. $(\sqrt[3]{12})^4$.
9. $(216^{\frac{1}{2}})^3$.
10. $(-3\sqrt{32})^2$.
11. $(a\sqrt[3]{a^2b^2x})^4$.
12. $(2\sqrt[5]{128})^2$.
13. $(-\sqrt{3a^{\frac{1}{2}}})^3$.
14. $(5\sqrt{2(a^2+4ax+4x^2)})^3$.
15. $(ab\sqrt{3ab})^5$.

67. Evolution of radicals.

Simplify the following:

1. $\sqrt[3]{(5\sqrt{5})}$.
2. $\sqrt[5]{(32\sqrt{32})}$.
3. $\sqrt{(6\sqrt[3]{6^4})}$.
4. $\sqrt[5]{(4\sqrt[3]{4^2})}$.
5. $\sqrt[3]{(\sqrt{27a^3})}$.
6. $\sqrt[5]{(\sqrt[3]{243a^5})}$.
7. $\sqrt[3]{(\sqrt{(a+b)^{12}})}$.
8. $\sqrt{(2\sqrt{216})}$.
9. $\sqrt[6]{(8\sqrt{a^{12}})}$.
10. $5\sqrt[5]{(\sqrt{8\times 128})}$.
11. $3\sqrt[3]{(a^2\sqrt{a^5})}$.
12. $\sqrt{(\sqrt{\sqrt{512x^8}})}$.
13. $\sqrt[3]{(\sqrt{(a+b)^3})}$.
14. $\sqrt{(a+b)^{\frac{8}{2}}}$.
15. $\sqrt[5]{(\sqrt{a-b)^6}})$.
16. $\sqrt{(\sqrt[3]{m^2n^2x^4})}$.
17. $\sqrt{(3\sqrt[3]{81})}$.
18. $[(a^5b^5)^{\frac{1}{2}}]^{\frac{1}{15}}$.

68. Rationalization.

Reduce the following to equivalent fractions having rational denominators:

1. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{2}}$.
2. $\dfrac{1-\sqrt{2}}{\sqrt{6}}$.
3. $\dfrac{5\sqrt{3}}{3\sqrt{5}}$.
4. $\dfrac{1}{\sqrt{2}}$.
5. $\dfrac{2}{\sqrt[4]{2}}$.
6. $\dfrac{8}{\sqrt[5]{64}}$.
7. $\dfrac{2}{\sqrt[4]{8}}$.
8. $\dfrac{1}{\sqrt[3]{2}}$.
9. $\dfrac{x}{\sqrt{y}}$.
10. $\dfrac{\sqrt{x-y}}{\sqrt{x+y}}$.
11. $\sqrt{\dfrac{a}{b}}$.
12. $\dfrac{1}{\sqrt{a^3}}$.
13. $\dfrac{\sqrt{5}}{\sqrt{10}}$.
14. $\dfrac{2}{3\sqrt[3]{a^2}}$.
15. $\dfrac{3}{\sqrt{12}}$.
16. $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$.
17. $\dfrac{2\sqrt{5}}{\sqrt{5}+\sqrt{3}}$.
18. $\dfrac{\sqrt{7}+\sqrt{2}}{2\sqrt{14}+9}$.
19. $\dfrac{3\sqrt{2}-1}{3\sqrt{2}+1}$.
20. $\dfrac{29}{11+3\sqrt{7}}$.
21. $\dfrac{17}{3\sqrt{7}+2\sqrt{3}}$.
22. $\dfrac{10\sqrt{6}-2\sqrt{7}}{3\sqrt{6}+2\sqrt{7}}$.
23. $\dfrac{2\sqrt{3}+3\sqrt{2}}{5+2\sqrt{6}}$.
24. $\dfrac{25\sqrt{3}-4\sqrt{2}}{7\sqrt{3}-5\sqrt{2}}$.
25. $\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$.
26. $\dfrac{\sqrt{9+x^2}-3}{\sqrt{9+x^2}+3}$.
27. $\dfrac{2\sqrt{a+b}+3\sqrt{a-b}}{2\sqrt{a+b}-\sqrt{a-b}}$.
28. $\dfrac{(3+\sqrt{5})(\sqrt{5}-2)}{5-\sqrt{5}}$.
29. $\dfrac{\sqrt{x+2}+\sqrt{x-2}}{\sqrt{x+2}-\sqrt{x-2}}$.
30. $\dfrac{1}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$.
31. $\dfrac{1}{\sqrt{3}+\sqrt{5}-\sqrt{2}}$.
32. $\dfrac{\sqrt{3}}{\sqrt{4+\sqrt{3}}}$.
33. $\dfrac{\sqrt{5}}{\sqrt{\sqrt{6}+\sqrt{2}}}$.

34. $\dfrac{7\sqrt{18}-3\sqrt{2}}{6\sqrt{3}-2\sqrt{12}+\sqrt{2}}$.

35. $\sqrt{\dfrac{1-\dfrac{\sqrt{2}}{2}}{1+\dfrac{\sqrt{2}}{2}}}$.

36. $\sqrt{\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}}$.

37. $\dfrac{3+\sqrt{6}}{5\sqrt{3}-2\sqrt{12}-\sqrt{32}+\sqrt{50}}$.

Simplify the following:

38. $\dfrac{2\sqrt{15}+8}{5+\sqrt{15}}-\dfrac{8\sqrt{3}-6\sqrt{5}}{5\sqrt{3}-3\sqrt{5}}$.

39. $\dfrac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}-\dfrac{x-\sqrt{x^2-a^2}}{x+\sqrt{x^2-a^2}}$.

40. $\dfrac{(3+\sqrt{3^2-2^2})^2-(3-\sqrt{3^2-2^2})^2}{(3-\sqrt{3^2-2^2})(3+\sqrt{3^2-2^2})}$.

41. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}-\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$.

42. $\left(\sqrt{\dfrac{a+x}{a-x}}-\sqrt{\dfrac{a-x}{a+x}}\right)\sqrt{\dfrac{a^3-x^3}{(a+x)^2-ax}}$.

43. $\dfrac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\dfrac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}$.

Find the values to three decimal places of the following:

44. $\dfrac{1}{\sqrt{2}-1}$.

45. $\dfrac{2-\sqrt{3}}{2+\sqrt{3}}$.

46. $\dfrac{2-\sqrt{3}}{3-\sqrt{2}}$.

47. $\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$.

48. $\dfrac{1}{\sqrt{8}+\sqrt{3}}+\dfrac{1}{\sqrt{8}-\sqrt{3}}$.

49. $\dfrac{7+3\sqrt{5}}{7-3\sqrt{5}}+\dfrac{7-3\sqrt{5}}{7+3\sqrt{5}}$.

What factors will rationalize each of the following?

50. $x-y^{\frac{1}{2}}$. **52.** $x^{\frac{2}{3}}+y^{\frac{2}{3}}$. **54.** $x^{\frac{3}{2}}-y^{\frac{2}{3}}$.

51. $1+2^{\frac{1}{3}}$. **53.** $2^{\frac{1}{2}}-4^{\frac{1}{3}}$. **55.** $x^{\frac{3}{4}}-y^{\frac{2}{3}}$.

69. Binomial Surds.

Extract the square roots of the following:

1. $14+6\sqrt{5}$. **9.** $15+2\sqrt{56}$.

2. $16-6\sqrt{7}$. **10.** $2\frac{1}{4}+\sqrt{5}$.

3. $6+\sqrt{20}$. **11.** $8+4\sqrt{3}$.

4. $5-\sqrt{24}$. **12.** $\frac{7}{9}+\frac{4}{3}\sqrt{\frac{1}{3}}$.

5. $101-28\sqrt{13}$. **13.** $\frac{7}{2}+\frac{3}{2}\sqrt{5}$.

6. $280+56\sqrt{21}$. **14.** $4-\sqrt{15}$.

7. $117-36\sqrt{10}$. **15.** $6-\sqrt{35}$.

8. $4\frac{1}{2}+2\sqrt{2}$. **16.** $a^2-2b\sqrt{a^2-b^2}$.

17. $2(a^2+b^2)+2\sqrt{a^4+a^2b^2+b^4}$.

18. $(a+b)^2-4(a-b)\sqrt{ab}$.

19. $3b^2+a^2+\sqrt{6a^2b^2+a^4}$.

20. $3x-1+2\sqrt{2x^2+x-6}$.

21. $2a+2\sqrt{a^2-x^2}$.

Simplify the following:

22. $\dfrac{\sqrt{2}+\sqrt{45}}{\sqrt{2}+\sqrt{7-2\sqrt{10}}}$.

23. $\dfrac{\sqrt{3+2\sqrt{2}}-\sqrt{2}}{\sqrt{3-2\sqrt{2}}+\sqrt{2}}$.

24. $\dfrac{\sqrt{5+2\sqrt{6}}-\sqrt{5-2\sqrt{6}}}{\sqrt{5+2\sqrt{6}}+\sqrt{5-2\sqrt{6}}}$.

25. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}-\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-\sqrt{2+\sqrt{3}}}$.

26. $\dfrac{1}{\sqrt{12-\sqrt{140}}}-\dfrac{1}{\sqrt{8-\sqrt{60}}}-\dfrac{2}{\sqrt{10+\sqrt{84}}}$

27. $\dfrac{1}{\sqrt{11-2\sqrt{30}}}-\dfrac{3}{\sqrt{7-2\sqrt{10}}}-\dfrac{4}{\sqrt{8+4\sqrt{3}}}$.

70. Radical Equations.

Solve the following equations:

1. $\sqrt{x^2+11}=x+1$.
2. $2\sqrt{x^2-64}=2x-8$.
3. $\sqrt{x+10}=1+\sqrt{x+1}$.
4. $\sqrt{x+5}-1=\sqrt{x}$.
5. $\sqrt{x-4}-\sqrt{x+11}=-3$.
6. $\sqrt{3x-11}+\sqrt{3x}=\sqrt{12x-23}$.
7. $\sqrt{x+14}-\sqrt{3x+10}=0$.
8. $\sqrt{25x-29}-\sqrt{4x-11}=3\sqrt{x}$.
9. $\sqrt{12x-5}+\sqrt{3x-1}-\sqrt{27x-2}=0$.
10. $\sqrt{x+4ab}=2a+\sqrt{x}$.
11. $(\sqrt{x+4})^{\frac{1}{3}}=\sqrt{2}$.
12. $\sqrt[4]{2x+11}=\sqrt{5}$.
13. $\sqrt[4]{x+10}=\sqrt{\sqrt{x+21}-1}$.

14. $\sqrt{x+9}+\sqrt{x-9}=4+\sqrt{34}$.

15. $\sqrt[3]{x^3+15x^2+25\sqrt{9x^2+64}}=x+5$.

16. $\sqrt{43+2\sqrt{3+\sqrt{6x}}}=7$.

17. $\sqrt{\sqrt{\sqrt{9x+3}+1}}=2$.

18. $[(x+a)^2+2ab+b^2]^{\frac{1}{2}}=b-a-x$.

19. $1+2\sqrt{x}=\sqrt{4x+\sqrt{16x+2}}$.

20. $\dfrac{\sqrt{1+x}+\sqrt{x-7}}{\sqrt{1+x}-\sqrt{x-7}}=2$.

21. $\sqrt{x}+\sqrt{9+x}=\dfrac{45}{\sqrt{9+x}}$.

22. $\dfrac{-1}{\sqrt{x+1}}+\dfrac{1}{\sqrt{x-1}}=\dfrac{1}{\sqrt{x^2-1}}$.

23. $2\sqrt{x}-\sqrt{4x-3}=\dfrac{1}{\sqrt{4x-3}}$.

24. $\dfrac{\sqrt{2+x}+\sqrt{2-x}}{\sqrt{2+x}-\sqrt{2-x}}=2$.

25. $\dfrac{\sqrt{x}+3}{\sqrt{x}+5}=\dfrac{2\sqrt{x}-1}{2\sqrt{x}+1}$.

26. $\dfrac{6\sqrt{x}-7}{\sqrt{x}-1}-5=\dfrac{7\sqrt{x}-26}{7\sqrt{x}-21}$.

27. $\dfrac{1}{x+\sqrt{x^2-50}}+\dfrac{1}{x-\sqrt{x^2-50}}=\dfrac{7}{25}$.

28. $\dfrac{1}{x+\sqrt{x^2-1}}+\dfrac{1}{x-\sqrt{x^2-1}}=6$.

29. $\dfrac{1}{1-x}+\dfrac{1}{1+\sqrt{x}}-\dfrac{1}{1-\sqrt{x}}=0$.

30. $(x + c\sqrt{4x + 2c})^{\frac{1}{2}} = c + \sqrt{x}.$

31. $\sqrt{\frac{x}{a} + \frac{c}{b}} + \sqrt{\frac{x}{a} - \frac{c}{b}} = \sqrt{\frac{4x}{a} - \frac{2c}{b}}.$

32. $(1 + x)^{\frac{1}{2}} + x^{\frac{1}{2}} = 2(1 + x)^{-\frac{1}{2}}.$

33. $(x + 11)^{\frac{1}{2}} + x^{\frac{1}{2}} = 55x^{-\frac{1}{2}}.$

34. $\dfrac{2x - 3}{(x - 2)^{\frac{1}{2}} + 1} = 2(x - 2)^{\frac{1}{2}} - 1.$

35. $\sqrt{ax - x} + \sqrt{a - x} = a\sqrt{1 - x}$

36. $\sqrt{1 + x + \sqrt{x}} + \sqrt{1 + x - \sqrt{x}} = a.$

37. $\sqrt{x + 3} + \sqrt{x - 6} = 3(1 + \sqrt{2}).$

38. $\sqrt{x + 2 + \sqrt{x^2 + 2}} + \sqrt{x - 2 + \sqrt{x^2 + 2}} = 2.$

39. $\sqrt{x - a} - \sqrt{x - b} = \sqrt{b} - \sqrt{a}.$

40. $\sqrt{x} + \sqrt{x - \sqrt{1 - x}} = 1.$

41. $x - 7 - \sqrt{49 + \sqrt{121x^2 + x^4}} = 0.$

42. $1 + \sqrt{1 + x} - \sqrt{1 + x + \sqrt{1 - x}} = 0.$

43. $\sqrt{1 + x} + \sqrt{1 + x + \sqrt{1 - x}} = \sqrt{1 - x}.$

44. $\sqrt[3]{x + 3} = \sqrt[6]{(x + 3)(3x^2 + 2x - 21)}.$

45. $\sqrt{a + \sqrt{a^2 + x}}(\sqrt{a - \sqrt{x}}) = \sqrt{x}.$

46. $\dfrac{5x - 1}{\sqrt{5x} + 1} = 1 + \dfrac{\sqrt{5x} - 1}{2}.$

47. $\dfrac{3x - 1}{\sqrt{3x} + 1} = 7\left(\dfrac{\sqrt{3x} - 1}{x - 5}\right).$

48. $\dfrac{\sqrt{x} - 1}{x - 1} = 1 - \dfrac{\sqrt{x}}{2\sqrt{x} - 5}.$

71. Imaginary Numbers.

Simplify:

1. $\sqrt{-4}$.
2. $\sqrt{-25}$.
3. $\sqrt{-144}$.
4. $\sqrt{-x^2}$.
5. $\sqrt{-(a-b)^2}$.
6. $\sqrt{-\frac{1}{9}}$.
7. $\sqrt{-\frac{m^4}{n^6}}$.
8. $\sqrt{-4a^6}$.
9. $(\sqrt{-1})^2$.
10. $(\sqrt{-1})^3$.
11. $(\sqrt{-1})^4$.
12. $(\sqrt{-1})^6$.
13. $(\sqrt{-1})^{16}$.
14. $(\sqrt{-1})^{38}$.
15. $\sqrt{-9}+\sqrt{-4}$.
16. $\sqrt{-36}-\sqrt{-144}+\sqrt{-121}$.
17. $\sqrt{-2}+\sqrt{-72}-\sqrt{-32}$.
18. $\sqrt{-84}+\sqrt{-\frac{3}{7}}$.
19. $(\sqrt{-2})(\sqrt{-2})$.
20. $\sqrt{-16}\times\sqrt{-9}$.
21. $\sqrt{-9a^2}\cdot\sqrt{-4b^4}$.
22. $\sqrt{-1}\cdot\sqrt{-4}\cdot\sqrt{25}$.
23. $\sqrt{-1}\cdot\sqrt{-28}\cdot\sqrt{-1}\ \sqrt{-32}$.
24. $-3\sqrt{-2}\cdot-2\sqrt{-3}$.
25. $\sqrt{-(x+y)}\cdot\sqrt{-(x-y)}$.
26. $(1-\sqrt{-2})\,(1+\sqrt{-2})$.
27. $(a+\sqrt{-1})(a-\sqrt{-1})$.
28. $(3+5\sqrt{-1})\,(4-7\sqrt{-1})$.
29. $\sqrt{-9}\div\sqrt{-3}$.
30. $a\div\sqrt{-a^2}$.
31. $-\sqrt{15}\div\sqrt{-3}$.
32. $-\sqrt{-12}\div-\sqrt{-3}$.

33. $-8\sqrt{-x^3} \div 4\sqrt{x}$.

34. $(3 + 2\sqrt{-1}) + (4 - 2\sqrt{-1}) + (7 + 3\sqrt{-1})$.

35. $\sqrt{-25} + \sqrt{-49} + \sqrt{-121} - \sqrt{-64}$.

36. $(5 + \sqrt{-16}) + (3 - \sqrt{-4}) + (8 + \sqrt{-4})$.

37. $(2\sqrt{-3} + 3\sqrt{-2})(4\sqrt{-3} - 5\sqrt{-2})$.

38. $(3\sqrt{-7} - 5\sqrt{-2})(3\sqrt{-7} + 5\sqrt{-2})$.

39. $(5 - \sqrt{-3} - \sqrt{-7})(5 + \sqrt{-3} + \sqrt{-7})$.

40. $(\sqrt{-3} - \sqrt{-2})^2$.

41. $(2\sqrt{-3} - 3\sqrt{-2})^2$.

42. $(\sqrt{-2} + \sqrt{-3})^3$.

43. $(\sqrt{-2} + \sqrt{-3} + \sqrt{-1})(\sqrt{-2} + \sqrt{-3} - \sqrt{-1})$.

44. $(\sqrt{-3} - \sqrt{2} + \sqrt{-2})(\sqrt{3} + \sqrt{-2} - \sqrt{2})$.

45. $(2\sqrt{-2} + \sqrt{-3}) \div (2\sqrt{-2} - \sqrt{-3})$.

46. $\dfrac{7 + \sqrt{-3}}{2 - \sqrt{-3}} + \dfrac{8 + 3\sqrt{-3}}{2 + \sqrt{-3}} - \dfrac{4(2 - \sqrt{-3})}{1 - \sqrt{-3}}$.

47. $\dfrac{3 + 2\sqrt{-1}}{2 - 5\sqrt{-1}} + \dfrac{3 - 2\sqrt{-1}}{2 + 5\sqrt{-1}}$.

48. $\dfrac{3\sqrt{-2} + 2\sqrt{-5}}{3\sqrt{-2} - 2\sqrt{-5}}$.

49. $(5\sqrt{-5} + \sqrt{-1})^2$.

50. $(2\sqrt{-3} - \sqrt{-1})^3$.

51. $\sqrt{-26 + 2\sqrt{-27}}$.

52. $\sqrt{2 - 4\sqrt{-2}}$.

53. $\sqrt{12\sqrt{-1} - 5}$.

54. $\sqrt{-3 - 3\sqrt{-48}}$.

55. $\sqrt{-8\sqrt{-1}}$.

56. $\sqrt{(ab)^2 - 1 + 2ab\sqrt{-1}}$.

72. Examinations in Radicals.

A

1. Simplify

(1) $5\sqrt{40}$. (3) $\sqrt[5]{4\sqrt{2}}$.

(2) $8\sqrt[3]{\frac{1}{8}}$. (4) $\sqrt[3]{3\sqrt{3}}$.

(5) $(a+2)\sqrt{2a^3 - 8a^2 + 8a}$.

(6) $(2\sqrt{72} - \sqrt{128})(5\sqrt{2} + \sqrt{8})$.

(7) $\dfrac{\sqrt{x-2} - \sqrt{x+2}}{\sqrt{x-2} + \sqrt{x+2}}$.

(8) $\dfrac{13}{\sqrt{37 + 20\sqrt{3}}}$.

(9) $(\sqrt{5} - \sqrt{2})(\sqrt{2}+1)(\sqrt{2}-1)(\sqrt{5} + \sqrt{2})$.

(10) $3\sqrt{32} + 2\sqrt{8} - \sqrt[3]{\frac{1}{2}} + \frac{1}{2}\sqrt{2} + \sqrt[4]{4}$.

2. Reduce to a common radical index $\sqrt{5}$, $\sqrt[3]{3}$, and $\sqrt[4]{2}$.

3. Solve the equation $\sqrt{x+7} + \sqrt{x+2} = 5$.

4. Find the value of $(1 + \sqrt{-1})^2 + (1 - \sqrt{-1})^2$.

B

1. Simplify:

(1) $2\sqrt{32}$. (3) $\frac{1}{2}\sqrt[3]{\frac{1}{4}}$.

(2) $5\sqrt{\frac{1}{5}}$. (4) $\sqrt{\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}}$.

(5) $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{2}+\sqrt{2+\sqrt{3}}} - \dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$.

(6) $\dfrac{\sqrt{a+b}+\sqrt{a-b}}{\sqrt{a+b}-\sqrt{a-b}}+\dfrac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}$.

2. Express with a rational denominator $\dfrac{1}{\sqrt{2}-\sqrt{3}-\sqrt{5}}$.

3. Solve the equation $\sqrt{x+14}+\sqrt{x+7}=7$.

4. If $x=1+2\sqrt{-1}$ and $y=2+\sqrt{-1}$, find the value of $\sqrt[3]{x^2+y^2}$.

C

1. Simplify

(1) $4\sqrt{147}-\dfrac{10}{\sqrt{3}}-3\sqrt{75}-2\sqrt{\frac{1}{3}}$.

(2) $5+3\sqrt{8}-7\sqrt{6}+3\sqrt{24}-\sqrt{31-10\sqrt{6}}$.

(3) $\sqrt[3]{\frac{3}{4}}\times\sqrt[3]{\frac{8}{9}}$.

(4) $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}-\dfrac{\sqrt{2}-1}{\sqrt{2}+1}$.

(5) $\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}$.

(6) $\dfrac{x+y\sqrt{-1}}{x-y\sqrt{-1}}+\dfrac{x-y\sqrt{-1}}{x+y\sqrt{-1}}$.

2. Square $5\sqrt{3}$, $ab\sqrt[3]{a^2b^2}$, and $\sqrt{(a^2-b^2)(a-b)}$.

3. Solve the equation $\sqrt{x+14}+\sqrt{x+2}=6$.

4. Solve the equation $\sqrt{x}+\sqrt{x+9}=\dfrac{36}{\sqrt{x}}$.

5. If $x=1+2\sqrt{-1}$, find the value of x^3-x^2+3x+5.

73. General Review of Algebra to Quadratics.

Exercises from College Entrance Examinations.

1. Simplify: $a - [2a - b - (3a - 2b - \overline{4a - 3b})]$.

2. Factor: $1 - 18x - 63x^2$; $(4ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2$.

3. Find the H. C. F. of $a^2 - 5ab + 4b^2$ and $a^3 - 5a^2b + 4b^3$.

4. Simplify: $\dfrac{x^2 - yz}{(x + y)(x + z)} + \dfrac{y^2 - xz}{(y + z)(y + x)} + \dfrac{z^2 - xy}{(z + x)(z + y)}$.

5. Find the square root of $x^{\frac{5}{3}} + x^{\frac{1}{3}} + 2x - 4x^{\frac{4}{3}} + 4x^{\frac{2}{3}}$.

6. Solve for x and y: $\dfrac{1}{3x} + \dfrac{1}{5y} = \dfrac{2}{9}$; $\dfrac{1}{5x} + \dfrac{1}{3y} = \dfrac{1}{4}$.

7. Simplify the following expressions:

(*a*) $(x^6)^{\frac{1}{3}} + 4\sqrt[4]{x^8}$;

(*b*) $(x^{\frac{2}{3}} \cdot x^{\frac{4}{5}})^{\frac{5}{11}}$;

(*c*) $\sqrt{75} + \sqrt{48} - \sqrt{243}$;

(*d*) $\sqrt{18} \times \sqrt{8}$;

(*e*) $\dfrac{x - 1}{x + 1}\sqrt{\dfrac{x + 1}{x - 1}}$.

8. Solve: $\sqrt{2x + 9} + \sqrt{3x - 15} = \sqrt{7x + 8}$.

9. A man has two sums of money at interest, one at 4%, and the other at 5%. Together they yield \$750 per year. If each sum yielded one per cent more interest, the man would have \$165 more income. How large are the sums of money?

10. Simplify: $\left(a^{-\frac{1}{2}}x^{\frac{1}{3}}\sqrt{ax^{-\frac{1}{3}}\sqrt[4]{x^{\frac{4}{3}}}}\right)^{\frac{1}{3}}$.

11. Two workmen complete some work in 20 days; if the first had worked twice as fast, and the second half as fast, they would have completed it in 15 days. How long would it have taken each alone to do the whole piece of work?

12. Express in simplest form: (*a*) $2\sqrt[3]{189} + 3\sqrt[3]{875} - 7\sqrt[3]{56}$; (*b*) $3\sqrt{147} - \frac{7}{8}\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{27}}$.

13. Extract the square root of $7 - 30\sqrt{-2}$.

14. (*a*) Simplify: $x^{3p+q} \cdot x^{p-4r}(x^2)^{q-2r} \div x^{4p-5r}$.

(*b*) Multiply: $x^n + x^{\frac{n}{2}} + 1$ by $x^{-n} + x^{-\frac{n}{2}} + 1$.

15. Simplify: $\left\{x - \frac{\frac{1}{y^{-1}}(x-y)}{x+y}\right\} \cdot \left\{1 - \frac{xy - y^2}{x^2}\right\} \cdot \left\{x - \frac{\frac{1}{y^{-2}}(x-y)}{x^2+y^2}\right\}$.

16. Simplify: $\frac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}} - \frac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}} + \frac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}$.

17. Which is greater $(\frac{1}{2})^{\frac{1}{2}}$ or $(\frac{2}{3})^{\frac{2}{3}}$? Prove your answer.

18. Solve the simultaneous equations:

$$\frac{1}{2(x+1)} + \frac{4}{3(y+1)} = 5, \quad \frac{1}{x+1} - \frac{1}{3(y+1)} = 1$$

and verify your solution.

19. Simplify the following expressions:

(*a*) $(a+b)^{\frac{1}{m}} \cdot (a+b)^{\frac{1}{n}} \cdot (a-b)^{\frac{1}{m}} \cdot (a-b)^{\frac{1}{n}}$;

(*b*) $72^{\frac{1}{3}} - 3(\frac{1}{3})^{\frac{1}{3}}$; (*c*) $\frac{5}{14}(\frac{2}{3})^{\frac{1}{3}} \div \frac{5}{21}(\frac{9}{4})^{\frac{1}{3}}$.

20. Factor: (*a*) $a^4 + a^2b^2 - b^2c^2 - c^4$.

(*b*) $a^2 - 4b^2 - 9c^2 + 12bc$.

21. Reduce to a fraction with a rational denominator:

$$\frac{\sqrt{3}}{2-\sqrt{3}} - \frac{2-\sqrt{-2}}{2+\sqrt{-2}}.$$

22. Solve: $\dfrac{\sqrt{3x}-\sqrt{3}}{\sqrt{2x}-\sqrt{2}} = \dfrac{\sqrt{x}+3}{\sqrt{x}+2}$.

23. Find the value of $\dfrac{9^{-2} \div 16^{\frac{7}{4}}}{81^{-\frac{3}{2}} \div 4^{\frac{3}{2}}}$.

24. Remove the parenthesis and brackets, writing the result as the product of two factors:

$(x-y)(x^2+xy)-(y-x)y^2+x^2(x-y)-y[x^2-(2x^2-y^2)]$.

25. Simplify: $\dfrac{a^{\frac{2}{5}}c^{-1}\sqrt{b}}{\sqrt[4]{b^3}} \cdot a^{-3}\sqrt[5]{ab^5}$.

26. Find the square root of:

$$\frac{a^2x^2}{9} + \frac{b^2}{25} + \frac{9x^4}{4} - ax^3 - \frac{3bx^2}{5} + \frac{2abx}{15}.$$

27. A and B buy stock, A buying twice as much as B. If A had invested \$1000 more and B \$1000 less, A would have invested three times as much as B. How much has each invested?

28. Simplify the following expressions:

(*a*) $\sqrt{a^2-x^2} + x^2(a^2-x^2)^{-\frac{1}{2}}$;

(*b*) $a\sqrt[3]{a^3d} + b\sqrt[3]{b^3d} - c\sqrt[3]{c^3d}$;

(*c*) $2\sqrt[3]{3} \times 2\sqrt{2}$; (*d*) $\left(\dfrac{64a^3}{27b^{-3}}\right)^{-\frac{1}{3}}$.

29. A has *a* hours to spare for an outing. How far can he ride with a friend at the rate of *b* miles an hour, and just consume the time remaining in walking back at the rate of *c* miles an hour?

30. (*a*) Multiply $\sqrt[4]{m^{-3}} - \sqrt{m^{-1}}\sqrt[4]{n} + \sqrt[4]{m^{-1}}\sqrt{n} - \sqrt[4]{n^3}$ by $m^{-\frac{1}{4}} + n^{\frac{1}{4}}$;

(*b*) Divide $\sqrt[3]{9a^4b^2c^{-5}}$ by $\sqrt{27a^3bc^{-\frac{10}{3}}}$.

31. Solve for y: $8^{\frac{1}{6}} + y^{-\frac{1}{3}} = \dfrac{7}{3 - \sqrt{2}}$.

32. Express as an integer or simple fraction:

$$12^\circ + 4\tfrac{1}{2} - 9^{-1} + \frac{1}{\sqrt[3]{-64}} + 27^{\frac{2}{3}}.$$

33. Two men charter a boat for a fixed sum. A wishes to send 80 passengers and B as many as will make his share of the cost \$90. Later B adds 40 to his list, and this raises his share to \$100. What was the contract price for the boat?

34. Which is the greatest, $\sqrt{\frac{4}{5}}$, $\sqrt[3]{\frac{8}{9}}$, or $\sqrt[4]{\frac{7}{4}}$?

35. Simplify: $\dfrac{\sqrt{\left(2^{\frac{1}{3}} + 2^{-\frac{1}{3}}\right)^2 - 4}}{4\left(2^{\frac{1}{3}} - 2^{-\frac{1}{3}}\right)} \times \dfrac{5 + \sqrt{21}}{5 - \sqrt{21}}$.

36. A and B are two stations 300 miles apart. Two trains start at the same time, one from A, the other from B, and travel to the opposite station. If the first train reaches B 9 hours after the trains meet, and the second train reaches A 4 hours after they meet, when do they meet, and what is the rate of each train?

37. Simplify: $\dfrac{\left\{9^n \cdot 3^2 \cdot \dfrac{1}{3^{-n}}\right\} - 27^n}{3^{3n} \cdot 9}$.

38. When $a = 2$, $b = 8$, $c = 4$, find the value of $\sqrt{\dfrac{a^2b^{-\frac{2}{3}}}{c^0}} + \sqrt{\dfrac{b}{ac^{-1}}}$ to two decimal places.

39. Simplify: $2(8)^{\frac{2}{3}} - \sqrt{3}\,(12)^{\frac{1}{2}} - 2(3)^0 + (a^{\frac{1}{3}}b^{-1})^3b^3 - \dfrac{1}{a^{-1}+b^{-1}}$.

40. Solve: $2\sqrt{x} + \sqrt{4x - \sqrt{16x-3}} = 1$.

41. Express in simplest radical form: $\dfrac{\sqrt{\frac{8}{2}} - \sqrt{\frac{8}{27}}}{\sqrt{2}\left(\sqrt{3} + \dfrac{1}{\sqrt{3}}\right)}$.

42. The middle digit of a number of three digits is 2. The sum of the digits is 6. If the digits are written in reverse order, the number will be multiplied by $1\frac{0}{1}\frac{7}{1}$. Find the number.

43. Divide: $(-25a^{-3}x^2\sqrt{2a^3yc^5})$ by $(-5a^{-4}x^n\sqrt[3]{2a^2yc^4})$

44. Find the value of x correct to three decimal places in the equation $\dfrac{3.7x}{.9} - \dfrac{2\frac{3}{4}x}{.02} = 1.72 + \dfrac{x}{.04}$.

45. Express in simplest radical form $\dfrac{1 - x(5+x^2)^{-\frac{1}{2}}}{x - \sqrt{5+x^2}}$.

46. Simplify: $\dfrac{\left(\dfrac{a}{27} \div \dfrac{a^{-2}}{8}\right)^{-\frac{2}{3}} - x^2}{\dfrac{3a^{-1}+2x}{2}}$.

47. Factor: $(a^2-4)^2 - (a+2)^2$; $2(x^3-1) + 7(x^2-1)$; $x^4 - 7x^2y^2 + y^4$.

48. Find the value of $\dfrac{1}{8^{-\frac{2}{3}}} - 3a^0 + 27^{-\frac{1}{3}} - 1^{\frac{8}{2}}$.

49. Solve for x and y the simultaneous equations:

$$\frac{2x-c}{a-2y} = \frac{a}{c}$$

$$\frac{2x-c}{a} = \frac{3x-y}{a+2c}.$$

50. A and B run a mile. At the first heat A gives B a start of 20 yards and beats him by 30 seconds. In the second heat A gives B a start of 32 seconds and beats him by $9\frac{5}{11}$ yards. Find the rate at which A runs.

51. Simplify: $\dfrac{3\sqrt{-5}+2\sqrt{-2}}{3\sqrt{-5}-2\sqrt{-2}}$; $\sqrt{5-12\sqrt{-1}}$.

52. Find the diameter of a circle circumscribed about a triangle whose sides are 5, 7, and 8, using the formula $d=\dfrac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}}$ where d is the diameter; a, b, and c represent the sides, and $s=\dfrac{a+b+c}{2}$. Find the result correct to two decimal places.

53. Find the value to two decimal places of

$$\sqrt{20}+3\sqrt{\tfrac{4}{5}}+3(\tfrac{5}{8})^{-\frac{1}{2}}-(\tfrac{32}{81})^{\frac{1}{4}}.$$

XV. QUADRATIC EQUATIONS

74. Incomplete Quadratic Equations.

Solve the following equations:

1. $\sqrt{2x+8}+2\sqrt{x+5}=2.$
2. $(x+1)^3-(x-1)^3=26.$
3. $x\sqrt{x^2+12}+x\sqrt{x^2+6}=3.$
4. $\sqrt{\frac{a}{x}}+\sqrt{\frac{x}{a}}=\sqrt{\frac{x}{b}}+\sqrt{\frac{b}{x}}.$
5. $x+\sqrt{a^2+x^2}=\frac{5a^2}{\sqrt{a^2+x^2}}.$
6. $\frac{5x+3\sqrt{2x^2+9}}{5x-3\sqrt{2x^2+9}}=19.$
7. $\frac{\sqrt{2x^2+1}-\sqrt{x^2-3}}{\sqrt{2x^2+1}+\sqrt{x^2-3}}=\frac{1}{2}.$
8. $\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}+\frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}=98.$
9. $\frac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}=\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}.$
10. $\sqrt{\frac{x^2-16}{x-3}}+\sqrt{x+3}=\frac{7}{\sqrt{x-3}}.$
11. $\frac{1}{m-\sqrt{m^2-x^2}}-\frac{1}{m+\sqrt{m^2-x^2}}=\frac{m}{x^2}.$
12. $\frac{x+a}{x-a}+\frac{x-a}{x+a}=\frac{3a+2b}{3a-2b}+\frac{3a-2b}{3a+2b}.$
13. $\frac{2x^2+1+x\sqrt{4x^2+1}}{2x^2+3+x\sqrt{4x^2+1}}=a.$

14. $\dfrac{x+a+2b}{x+a-2b} = \dfrac{b-2a+2x}{b+2a-2x}$.

15. $\sqrt{1-x+x^2} - \sqrt{1+x+x^2} = m$.

75. Complete Quadratic Equations.

Solve the following equations:

1. $x^2 + 9x = 22$.
2. $x^2 - 6x = 27$.
3. $x^2 - 8x = -12$.
4. $x^2 - 4x = 5$.
5. $x^2 + 5x = -6$.
6. $x^2 - 3x = 130$.
7. $3x^2 + 7x = 6$.
8. $5x^2 + 9x = 2$.
9. $5x - 6x^2 + 1 = 0$.
10. $35x^2 - 3x = 2$.
11. $17x + 2 = -8x^2$.
12. $4x^2 + 8x = -3$.
13. $6x^2 + 8x = 8$.
14. $8x - 15x^2 = 1$.
15. $6x - 2x^2 = 4$.
16. $10x^2 + 12x + 2 = 0$.
17. $4x^2 - 2x = 2$.
18. $9x^2 + 9x = -2$.

76. Numerical Fractional Equations.

Reduce to the proper form and solve the following:

1. $\dfrac{x-1}{x} + \dfrac{x}{x-1} = \dfrac{5}{2}$.
2. $\dfrac{x+1}{x+2} + \dfrac{x+2}{x+1} = \dfrac{26}{5}$.
3. $\dfrac{x}{2x-1} + \dfrac{8-x}{x+5} = \dfrac{13}{6}$.
4. $\dfrac{5-x}{x-3} + \dfrac{4}{(x-1)(x-3)} = 0$.
5. $\dfrac{1}{(x-1)(x-3)} - 6 = \dfrac{2}{x-1} - \dfrac{3}{x-3}$.
6. $\dfrac{4x-3}{3x-7} = 3 + \dfrac{2x-3}{x-1}$.
7. $\dfrac{5x+2}{2x+5} + \dfrac{1}{20} = \dfrac{2x+5}{3(x+2)}$.
8. $\dfrac{2}{3(x-4)} = \dfrac{5}{4(x-2)} + \dfrac{1}{4}$.

9. $\frac{x^3+1}{x+1}+\frac{x^4-1}{x^2+1}=6.$

10. $\frac{8-x}{2}-\frac{2x-11}{x-3}=\frac{x-2}{6}.$

11. $\frac{x}{x^2-2x-15}-\frac{7\frac{1}{2}}{x^2+2x-35}=\frac{1}{x^2+10x+21}.$

12. $\frac{1}{x}+\frac{1}{x+4}=\frac{1}{x+1}+\frac{1}{x+2}.$

13. $\frac{x}{x+1}+\frac{x+1}{x+2}+\frac{x+2}{x+3}=3.$

14. $\frac{x}{x+1}+\frac{x+1}{x+2}=\frac{x-2}{x-1}+\frac{x-1}{x}.$

15. $\frac{x-1}{x+1}+\frac{x+1}{x-1}=\frac{5x}{x^2-1}.$

16. $\frac{x^2-5x}{x+3}=x-3+\frac{1}{x}.$

17. $\frac{4}{x+1}+\frac{5}{x+2}=\frac{12}{x+3}.$

77. Literal Equations.

Reduce to the proper form and solve:

1. $4x^2+4ax=b^2-a^2.$
2. $x^2+2ab=2ax+b^2.$
3. $x^2-2ax+4ab=2bx.$
4. $x^2-2ax+8x=16a.$
5. $3x^2-2ax-bx=0.$
6. $(b-c)x^2+(c-a)x+a-b=0.$
7. $(a+b)x^2+cx-a-b-c=0.$
8. $abx^2-(a^2+b^2)x+ab=0.$
9. $(b^2-a^2)(x^2+1)=2x(a^2+b^2).$
10. $x^2+2x(b-c)+c^2=2bc.$
11. $(a^2-b^2)(x^2-1)=4abx.$
12. $x^2-a^2x-b^2x=ab^3-a^3b.$

13. $\frac{x}{a} + \frac{a}{x} = \frac{a}{b} + \frac{b}{a}$.

14. $x + \frac{1}{x} = a + \frac{1}{a}$.

15. $\frac{a}{x-a} + \frac{b}{x-b} = \frac{2c}{x-c}$.

16. $\frac{(a-x)^2 + (b-x)^2}{(a-x)(b-x)} = \frac{5}{2}$.

17. $\frac{x-a}{x-b} + \frac{x-b}{a-x} = \frac{m}{n}$.

18. $\frac{mx}{mx+a} + \frac{nx}{nx+a} = \frac{2px}{px+a}$.

19. $\frac{(x+b)^2}{a^2} - \frac{(x+a)^2}{b^2} = \frac{b^4 - a^4}{a^2b^2}$.

20. $(a+b)(ax+b)(a-bx) = (a^2x - b^2)(a+bx)$.

21. $\frac{(x-a+b)^2}{(x+a-b)^2} = \left(1 - \frac{2x}{a}\right)^2 + \left(1 + \frac{2x}{b}\right)^2 - 9$.

22. $\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{c+a} + \frac{1}{c+b}$.

23. $\frac{x}{x+a} + \frac{x}{x+b} = \frac{c}{c+a} + \frac{c}{c+b}$.

24. $\frac{a+c(a+x)}{a+c(a-x)} = \frac{a}{a-2cx} - \frac{a+x}{x}$.

78. Formulas Which Involve Quadratic Equations.

1. Solve for t, $S = \frac{1}{2}at^2$.

2. Solve for n, $S = \frac{n}{2}[2a + (n-1)d]$.

3. Solve for R, $V = \frac{1}{3}\pi R^2 h$. Find the value of R, if $V = 132$, and $h = 14$.

4. Solve for t, $S = vt - 16t^2$.

5. Solve for v, $m=\frac{gv^2}{c}$. Find v correct to three figures if $m = 12$, $c = 3.3$, $g = 15$.
6. Solve for R, $T = 2\pi R[H + R]$. Find R, if $T = 352$ and $H = 10$.
7. Solve for t, $S = V_0t + \frac{1}{2}gt^2$. Find V_0 if $t = 5$, $g = 32$, and $S = 600$.
8. In the formula of number **7** find t if $S = 200$, $g = 32$, and $V_0 = 10$.

79. Equations Which Contain Radicals.

Reduce to the proper form and solve:

1. $2\sqrt{x+1} = \sqrt{x+8} + 2$.
2. $3\sqrt{x} = 16 - \sqrt{x-24}$.
3. $\sqrt{x+5} + \sqrt{x+16} = \sqrt{x+101}$.
4. $2\sqrt{x+3} = 4 + \sqrt{x-2}$.
5. $3\sqrt{x+20} = \sqrt{8x+9} + 8$.
6. $\sqrt{x+9} + \sqrt{x-7} = \sqrt{x-12}$.
7. $\sqrt{2x+7} + \sqrt{3x-18} = \sqrt{7x+1}$.
8. $\sqrt{4x^2+6x-9} - 2\sqrt{x^2-x-4} = 3$.
9. $2\sqrt{x+95} - 5 = \sqrt{5x+200}$.
10. $\sqrt{x+6} + \sqrt{x+13} = \sqrt{2x+43}$.
11. $5\sqrt{x+2} + 2\sqrt{x+7} = 4\sqrt{x+14}$.
12. $\sqrt{\sqrt{\sqrt{9x}+3}-1} = \sqrt[4]{x}$.
13. $\sqrt[3]{\sqrt{25-x} - \sqrt{x-12}} = \sqrt[6]{x-15}$.
14. $\sqrt{3x^2-7x-30} - \sqrt{2x^2-7x-5} = x - 5$.
15. $\sqrt{(x-3)(2x-3)} + \sqrt{(x-1)(2x-5)} = \sqrt{2}$.
16. $\sqrt{2x+4} - \sqrt{\frac{x}{2}+6} = 1$.

17. $\sqrt{3+x}+\sqrt{3-x}=2\sqrt{x}$.

18. $2\sqrt{x-2}=\sqrt[4]{64(x-2)^3}$.

19. $12\sqrt{\frac{x}{2}}+5\sqrt{\frac{2}{x}}=26\frac{1}{2}$.

20. $\frac{a-\sqrt{2ax-x^2}}{a+\sqrt{2ax-x^2}}=\frac{x}{a-x}$.

21. $\frac{x+\sqrt{12a-x}}{x-\sqrt{12a-x}}=\frac{\sqrt{a}+1}{\sqrt{a}-1}$.

22. $\frac{x^3+1}{x^2-1}=x+\frac{\sqrt{6}}{\sqrt{x}}$.

23. $\frac{\sqrt{4x}+1}{\sqrt{4x}-1}=\frac{\sqrt{x+3}+\sqrt{x}}{\sqrt{x+3}-\sqrt{x}}$.

24. $x+\sqrt{x^2+5x+6}=2\sqrt{x+2}+2\sqrt{x+3}$.

25. $\sqrt{a+x}+\sqrt{b+x}=\sqrt{a+b+2x}$.

26. $\sqrt{a-x}+\sqrt{b-x}=\sqrt{a+b-2x}$.

27. $\sqrt{ax+b^2}+\sqrt{bx+a^2}=a-b$.

28. $\sqrt{a-x}+\sqrt{b+x}=\sqrt{2(a+b)}$.

80. Problems Producing Quadratic Equations of One Unknown Quantity.

1. Find two consecutive numbers whose product is 462.

2. The difference of the cubes of two consecutive numbers is 217. What are the numbers?

3. A rectangular field contains two acres, and its length is 32 rods greater than its breadth. What are its length and breadth?

4. What is the price of eggs per dozen when one more for six cents reduces the price per dozen 12 cents?

5. A man buys a certain number of pounds of meat for \$8. If he had paid two cents per pound more, he would have got 20 pounds less for his money. How many pounds did he buy?

6. The dimensions of a picture inside the frame are 12 by 16. What is the width of the frame if its area is 288 square inches?

7. The area of a rectangle is 18 inches less than twice the area of a square. The rectangle is 7 inches longer than the square, and a side of the latter equals the breadth of the rectangle. Find the side of the square.

8. An open box is made from a square piece of tin by cutting out a 6-inch square from each corner and turning up the sides; the box contains 150 cu. in. Find the area of the original square.

9. If I had bought 18 fewer articles for \$336, each would have cost \$6 more. Find the price of each and the number of articles bought.

10. A piece of property is sold for \$29.76 at a per cent of profit equal to the cost in dollars. What did it cost?

11. A man bought a number of pieces of cloth for \$200, and sold them at \$8.32 per piece, gaining as much on the whole number as one piece cost him. What was the number of pieces?

12. A man having 7 miles to walk proceeds one mile at a certain rate per hour, and then completes the distance at a rate one mile per hour faster. He finds that he has been half an hour less upon the road than he would have been had the original rate been unchanged. How much time did he use in walking the 7 miles?

13. Two couriers start at the same time for a place 90 miles distant. The first travels one mile per hour faster than the second, and reaches the end of the journey one hour sooner. What was the rate of each?

14. A laborer digs two trenches, one 40 feet longer than the other. He receives for labor upon each as many cents per foot as it is feet in length, and for the whole work \$136. What is the length of each trench?

15. The difference between the hypotenuse and the other two sides of a triangle is respectively 8 and 4 feet. Find the other two sides.

16. A certain number consists of two digits whose product is 35, and if 18 is subtracted from the number the order of its digits will be inverted. What is the number?

17. A cistern can be filled by two pipes running together in $3\frac{3}{4}$ hours, but the larger one will fill it alone in four hours less time than the smaller one. What time is required by each to fill it alone?

18. The cost of a dinner was \$60, which was to have been divided equally among a certain number of men; but five of them failed to be present, and consequently the others were obliged to pay one dollar each more than they would otherwise have done. What was the original number?

19. A crew whose rate in still water is 6 miles an hour went down a stream a distance of 8 miles and returned in 4 hours 48 minutes from the time they started. What was the rate of the stream?

20. A man sold a number of barrels of apples for \$50. If he had sold 5 more barrels for 50 cents less a barrel, he would have received the same amount. Find the number of barrels and the price per barrel.

21. A man worked a number of days and earned \$75. If he had received 50 cents more per day, he would have earned the same amount in 5 days less. How many days did he work?

22. A certain number of bolts can be bought for a dollar. If 10 more could be bought for a dollar, the price would be half a cent less per dozen. What is the price per dozen?

23. A broker sells certain railroad shares for \$3240. A few days later, the price having fallen \$9 per share, he buys for the same sum five more shares than he had sold. Find the price and the number of shares transferred on each day.

24. An automobile runs 10 miles per hour faster than a bicycle and it takes the automobile 6 hours longer to run 255 miles than it does the bicycle to run 63 miles. What is the rate of each?

25. A and B run a half mile race. A, who is faster than B by half a yard per second, allows B a start of 15 seconds and beats him by 5 yards. Find their respective rates in yards per second. [1 mile = 1760 yards.]

26. A man paid \$300 for a drove of sheep. By selling all but 10 of them at a profit of \$2.50 each, he received the amount paid for all the sheep. How many sheep were there?

27. A man bought a number of cattle which cost him in all \$672. If each head had cost him \$4 less, he would have been able to buy 3 more. How many did he buy and at what price?

28. The circumference of the rear wheel of a carriage is 2 feet greater than the circumference of the front wheel. The front wheel makes 64 more revolutions than the rear wheel in traveling 3496 feet. What is the circumference of each wheel?

29. A man drives to a certain place at the rate of $8\frac{1}{2}$ miles an hour. He returns by a road that is $2\frac{1}{4}$ miles longer at the rate of 9 miles an hour and takes 10 minutes longer than in going. How long is each road?

30. A park is 120 rods long and 80 rods wide. It is decided to double the area of the park, still keeping it rectangular, by adding strips of equal width to one end and one side. Find the width of the strip.

31. Telegraph poles are placed at equal intervals along a certain railroad. In order that there should be two less poles per mile it would be necessary to increase the distance between consecutive poles by 24 feet. Find the number of poles to a mile.

32. A person drew a quantity of wine from a full vessel containing 81 gallons, and filled up the vessel with water. He then drew from the mixture the same amount that he drew the first time, and found that there remained in the vessel 64 gallons of pure wine. How much did he draw each time?

33. A man invests \$4000 at a certain rate of interest. At the end of one year he withdraws \$100 from the amount due, and leaves the remainder, both principal and interest, upon interest at the same rate. At the end of the second year he finds the amount due him to be \$4388.40. What was the rate of interest?

34. A man buys a certain number of \$100 shares for \$5100, when they are at a certain discount. He keeps ten shares, and sells the remainder for \$6500 when the premium is double the original discount. How many shares did he buy?

35. A man walks 6 miles into the country. On his return he walks $22\frac{1}{2}$ minutes, and is then detained the same length of time; he now finds that in order to use no more time in returning than in going out, he must for the remainder of the distance increase his speed two miles per hour. What was his ordinary rate?

81. Equations in the Quadratic Form.

Reduce to the quadratic form, if necessary, and solve the following:

1. $x^4 - 5x^2 + 4 = 0.$
2. $x^4 - 35x^2 + 216 = 0.$
3. $3x^{\frac{2}{3}} = 4x^{\frac{1}{3}} + 4.$
4. $3x + 2x^{\frac{1}{2}} - 1 = 0.$
5. $x^{\frac{1}{4}} + 5x^{\frac{1}{2}} = 22.$
6. $3x^{\frac{3}{2}} - 4x^{\frac{3}{4}} = 7.$
7. $6x^{\frac{1}{2}} = 5x^{-\frac{1}{2}} - 13.$
8. $2x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} = 5.$
9. $x^{\frac{3}{2}} + 8x^{\frac{1}{2}} = 9x.$
10. $x^{\frac{1}{2}} - 3 + 2x^{-\frac{1}{2}} = 0.$
11. $x^{\frac{3}{4}} - 2x^{\frac{1}{2}} + x = 0$
12. $x^{-2} - 2x^{-1} = 8.$
13. $x^{-4} - 10x^{-2} = -9.$
14. $x^{-1} + 3 = 4x^{-\frac{1}{2}}.$
15. $x^{-\frac{3}{2}} + 8 = 9x^{-\frac{3}{4}}.$
16. $6x^{\frac{3}{4}} = 7x^{\frac{1}{4}} - 2x^{-\frac{1}{4}}.$
17. $x^{2n} + 4x^n = 5.$
18. $x^{\frac{2}{n}} - 5x^{\frac{1}{n}} = -6.$
19. $3x^{\frac{1}{2n}} - x^{\frac{1}{n}} = 2.$
20. $2(\sqrt{x} - 3)^2 - 3 = \sqrt{x}.$
21. $27\sqrt[5]{x^{-6}} - 26\sqrt[5]{x^{-3}} = 1.$
22. $(x - 5)^2 + 5(x - 5) = 50.$
23. $(2x^2 + 1)^2 + (2x^2 + 1) = 90.$
24. $x + 1 + \sqrt{x + 1} = 6.$
25. $(x^2 - x)^2 - 8(x^2 - x) = -12.$
26. $(x^2 + x)^2 - 22(x^2 + x) = -40.$
27. $\left(\frac{x^2 + 1}{x}\right)^2 + 4\left(\frac{x^2 + 1}{x}\right) = 12.$
28. $x + 5 + 2\sqrt{x + 5} = 15.$
29. $x - 2 + 3\sqrt{x - 2} = 0.$
30. $5(x + 6) + 5\sqrt{x + 6} = 60.$
31. $6x + 3 = 3\sqrt{6x + 3}$

32. $\sqrt[4]{x-1} + 2\sqrt{x-1} - 1 = 0.$

33. $\sqrt[3]{7x+6} - 4\sqrt[6]{7x+6} + 4 = 0.$

34. $x^2 - 4x + 2\sqrt{x^2 - 4x + 7} = 1.$

35. $x^2 - 6x + 12 + \sqrt{x^2 - 6x + 9} = 5.$

36. $x^2 + 5x + 4 = 5\sqrt{x^2 + 5x + 28}.$

37. $2x^2 - 2x + 6\sqrt{x^2 - x + 7} = 22.$

38. $2x^2 + 6x = 226 - \sqrt{x^2 + 3x - 8}.$

39. $9\sqrt{x^2 - 9x + 28} + 9x = x^2 + 36.$

40. $x^2 + 5x - 10 + 2\sqrt{x^2 + 5x + 2} = 12.$

41. $x^2 - 4x + 6 + 4\sqrt{x^2 - 7x + 11} = 3x.$

42. $3x^2 - 7 + 3\sqrt{3x^2 - 16x + 21} = 16x.$

43. $25x^2 - 20ax + 2a\sqrt{25x^2 - 20ax + 3a^2} = 0.$

44. $(x-3)^2 - 13 = -\sqrt{x^2 - 6x + 16}.$

45. $9(x+a)^{\frac{4}{3}} - 22b^2(x+a)^{\frac{2}{3}} + 8b^4 = 0.$

46. $x^2 - x + 3\sqrt{2x^2 - 3x + 2} = \frac{x}{2} + 7.$

47. $\sqrt{3x-2} + 6 = 5\sqrt[4]{3x-2}.$

48. $(x+5)(x-2) + 3\sqrt{x^2 + 3x} = 0.$

49. $\frac{3x-2}{2} + \sqrt{2x^2 - 5x + 3} = \frac{(x+1)^2}{3}.$

50. $\sqrt{x^2 - 2x + 9} - \frac{x^2}{2} = 3 - x$

51. $4a\sqrt{x^2 - 4ax + 9a^2} = 12a^2 + 4ax - x^2.$

52. $x^4 + 2x^3 - 31x^2 - 32x + 60 = 0.$

53. $x^4 + 8x^3 + 24x^2 + 32x + 16 = 0.$

54. $x^4 + 4x^3 + 5x^2 + 2x = 240.$

55. $x^4 - 6x^3 + 10x^2 - 3x = 2$

56. $x^4 - 8x^3 + 12x^2 + 16x - 12 = 0.$

XVI. SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS

Equations of Two Unknown Quantities

Case I

82. When one equation is of the first degree and the other of the second.

Solve the following equations:

1. $\begin{cases} x + y = 5. \\ x^2 + 2y^2 = 22. \end{cases}$

2. $\begin{cases} x - y = 3. \\ xy = 10. \end{cases}$

3. $\begin{cases} x + y = 1\frac{5}{4}. \\ x - y = xy. \end{cases}$

4. $\begin{cases} x + 2y = 7. \\ \frac{3}{x} + \frac{6}{y} = 5. \end{cases}$

5. $\begin{cases} 2x + y = 16. \\ xy + y^2 = 66. \end{cases}$

6. $\begin{cases} x + y = 5. \\ \frac{1}{x} + \frac{1}{y} = \frac{5}{6}. \end{cases}$

7. $\begin{cases} \frac{x}{y} + \frac{y}{x} = \frac{5}{2}. \\ x - y = -2. \end{cases}$

8. $\begin{cases} 3x + 2y = 5. \\ 4x^2 + 9y^2 = 13. \end{cases}$

9. $\begin{cases} x^2 - y^2 = 5. \\ x + y = 5. \end{cases}$

10. $\begin{cases} x + y = -1. \\ x^2 + y^2 = 25. \end{cases}$

11. $\begin{cases} x^2 - y^2 = 24. \\ x - 2y = 3. \end{cases}$

12. $\begin{cases} x + y = 8. \\ xy = 12. \end{cases}$

13. $\begin{cases} x^2 - y^2 = 16. \\ 5x + y = 28. \end{cases}$

14. $\begin{cases} x - y = 1. \\ 7x^2 + 5xy - 6y^2 = 32. \end{cases}$

15. $\begin{cases} 2x - 3y = -2. \\ x^2 + 3xy + y^2 = 20. \end{cases}$

16. $\begin{cases} \frac{x}{2} + \frac{y}{5} = 5. \\ \frac{2}{x} + \frac{5}{y} = \frac{5}{6}. \end{cases}$

CASE II

83. When the equations are both symmetrical, or in forms which can be solved in the same manner as symmetrical equations.

Solve the following equations:

1. $\begin{cases} x^2 + y^2 = 26. \\ xy = 5. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 13. \\ x + y = 5. \end{cases}$

3. $\begin{cases} x - y = 4. \\ 2xy = 42. \end{cases}$

4. $\begin{cases} x + y = 10. \\ \sqrt{xy} = 3. \end{cases}$

5. $\begin{cases} x^2 + xy + y^2 = 49. \\ x + y = 8. \end{cases}$

6. $\begin{cases} x^2 - xy + y^2 = 211. \\ xy = 15. \end{cases}$

7. $\begin{cases} x - y = 2. \\ x^3 - y^3 = 98. \end{cases}$

8. $\begin{cases} x + y = 1. \\ x^3 + y^3 = 61. \end{cases}$

9. $\begin{cases} x - y = 37. \\ x^{\frac{1}{3}} - y^{\frac{1}{3}} = 1. \end{cases}$

10. $\begin{cases} x + y = 35. \\ x^{\frac{1}{3}} + y^{\frac{1}{3}} = 5. \end{cases}$

11. $\begin{cases} x^3 - y^3 = 98. \\ 3xy(x - y) = 90. \end{cases}$

12. $\begin{cases} x - y = 2. \\ x^2y - xy^2 = 30. \end{cases}$

13. $\begin{cases} x^2y + xy^2 = 30. \\ x^3 + y^3 = 35. \end{cases}$

14. $\begin{cases} x^2 + xy + y^2 = 13. \\ x^4 + x^2y^2 + y^4 = 91. \end{cases}$

15. $\begin{cases} \frac{1}{x^2} + \frac{1}{y^2} = 13. \\ \frac{1}{xy} = 6. \end{cases}$

16. $\begin{cases} x^2 + xy + y^2 = 68\frac{1}{4}. \\ x - \sqrt{xy} + y = 6\frac{1}{2}. \end{cases}$

17. $\begin{cases} x - y = 1. \\ \frac{x}{y} - \frac{y}{x} = \frac{5}{6}. \end{cases}$

18. $\begin{cases} x^4 + y^4 = 881. \\ xy = 20. \end{cases}$

19. $\begin{cases} \frac{1}{x^3} + \frac{1}{y^3} = \frac{7}{2}. \\ \frac{1}{x} + \frac{1}{y} = 2. \end{cases}$

20. $\begin{cases} xy(x + y) = 30. \\ \frac{x + y}{xy} = \frac{5}{6}. \end{cases}$

Case III

84. When both equations are homogeneous quadratics.

Solve the following equations:

1. $\begin{cases} y^2 + xy = 6. \\ x^2 + 3xy = 7. \end{cases}$

2. $\begin{cases} x^2 - 3xy = 10. \\ 4y^2 - xy = -1. \end{cases}$

3. $\begin{cases} x^2 + xy = 24. \\ 2y^2 + 3xy = 32. \end{cases}$

4. $\begin{cases} x^2 - 4y^2 = 9. \\ xy + 2y^2 = 3. \end{cases}$

5. $\begin{cases} x^2 + xy = 12. \\ xy - 2y^2 = 1. \end{cases}$

6. $\begin{cases} x^2 - xy + y^2 = 21. \\ y^2 - 2xy = -15. \end{cases}$

7. $\begin{cases} x^2 + xy - 6y^2 = 21. \\ xy - 2y^2 = 4. \end{cases}$

8. $\begin{cases} x^2 + 2y^2 = 22. \\ 3y^2 - xy - x^2 = 17. \end{cases}$

9. $\begin{cases} x^2 + xy + 4y^2 = 6. \\ 3x^2 + 8y^2 = 14. \end{cases}$

10. $\begin{cases} 2x^2 - 3xy + 5y^2 = 64. \\ 3x^2 + xy + y^2 = 36. \end{cases}$

11. $\begin{cases} x^2 + xy = a. \\ y^2 + xy = b. \end{cases}$

12. $\begin{cases} ay^2 + bxy - b = 0. \\ bx^2 + axy - a = 0. \end{cases}$

85. Miscellaneous Exercises.

Solve the following equations:

1. $\begin{cases} y^2 + xy = 15. \\ x^2 + xy = 10. \end{cases}$

2. $\begin{cases} x + y = -3. \\ \frac{1}{x} + \frac{1}{y} = \frac{1}{6}. \end{cases}$

3. $\begin{cases} x + \frac{1}{y} = 3. \\ y + \frac{1}{x} = \frac{12}{5}. \end{cases}$

4. $\begin{cases} ay^2 + bxy = b. \\ bx^2 + axy = a. \end{cases}$

5. $\begin{cases} x^2 + 3xy = 28. \\ xy + 4y^2 = 8. \end{cases}$

6. $\begin{cases} x + xy + y = 29. \\ x^2 + xy + y^2 = 61. \end{cases}$

7. $\begin{cases} \frac{x}{a} + \frac{y}{b} = 1. \\ \frac{a}{x} + \frac{b}{y} = 4. \end{cases}$

8. $\begin{cases} 5xy = 84 - x^2y^2. \\ x - y = 6. \end{cases}$

9. $\begin{cases} \frac{1}{x} + \frac{1}{y} = 2. \\ xy + \frac{1}{x} + \frac{1}{y} = 8. \end{cases}$

10. $\begin{cases} x - y = 2. \\ \sqrt{x} + \sqrt{y} = 2. \end{cases}$

11. $\begin{cases} x(x - y) = 3. \\ y(y - x) = -2. \end{cases}$

12. $\begin{cases} x + y = a. \\ x^2 + y^2 = b^2. \end{cases}$

13. $\begin{cases} \frac{x + y}{1 - xy} = 3. \\ \frac{x - y}{1 + xy} = \frac{1}{3}. \end{cases}$

14. $\begin{cases} .2x^2 - xy = -.742. \\ x + .1y = .82. \end{cases}$

15. $\begin{cases} \frac{25x}{y} = \frac{4y}{x}. \\ xy + x + y = 17. \end{cases}$

16. $\begin{cases} (x + y)^2 + x + y = 56. \\ xy = 10. \end{cases}$

17. $\begin{cases} x^2y + xy^2 = 30. \\ xy + y^2 = 10. \end{cases}$

18. $\begin{cases} x^4 + y^2 = 25. \\ x^2 + y = 7. \end{cases}$

19. $\begin{cases} x^4 + y^4 = 706. \\ x + y = 8. \end{cases}$

20. $\begin{cases} x^4 + y^4 = 272. \\ x - y = 2. \end{cases}$

21. $\begin{cases} 2x^2 - 3x + 4y = 14. \\ (x + y)^2 - (x + y) = 20. \end{cases}$

22. $\begin{cases} x^2 + y^2 = 13. \\ x + y + xy = 11. \end{cases}$

23. $x + y + 3\sqrt{x + y} = x^2 + y^2 = 10.$

24. $\begin{cases} x^{-2} + y^{-2} = 13. \\ 6xy = 1. \end{cases}$

25. $\begin{cases} x^{-1} + y^{-1} = 5. \\ (x + 1)^{-1} + (y + 1)^{-1} = \frac{17}{12}. \end{cases}$

26. $\begin{cases} x^3 - y^3 = 7xy. \\ x - y = 2. \end{cases}$

27. $\begin{cases} a + ar^3 = 56. \\ ar + ar^2 = 24. \end{cases}$

28. $\begin{cases} x^2 + 7xy + 12y^2 = 0. \\ x^2 - (y - 2)^2 = 0. \end{cases}$

29. $\begin{cases} x - \sqrt{xy} + y = 13. \\ x^2 + xy + y^2 = 273. \end{cases}$

30. $\begin{cases} 20 = a + (n - 1)2. \\ 108 = \frac{n}{2}(a + 20). \end{cases}$

31. $\begin{cases} x^2 - y^2 = 1. \\ xy - 2y + x = 2. \end{cases}$

32. $\begin{cases} x^2 - y^2 = 8ab. \\ xy = a^2 - 4b^2. \end{cases}$

33. $\begin{cases} x^{-3} + y^{-3} = 152. \\ x^{-1} + y^{-1} = 8. \end{cases}$

34. $\begin{cases} x + y = 25. \\ \sqrt{x} + \sqrt{y} = 7. \end{cases}$

35. $\begin{cases} x^{\frac{2}{3}} + y^{\frac{2}{3}} + 4(x^{\frac{1}{3}} + y^{\frac{1}{3}}) = 26. \\ x^{\frac{1}{3}} y^{\frac{1}{3}} = 3. \end{cases}$

36. $\begin{cases} x - y = \sqrt{x} + \sqrt{y}. \\ x^{\frac{3}{2}} - y^{\frac{3}{2}} = 61. \end{cases}$

86. Equations containing more than two unknown quantities.

Solve the following equations:

1. $\begin{cases} xy = 24. \\ xz = 32. \\ yz = 12. \end{cases}$

2. $\begin{cases} x^2 + (5 - y)^2 = z^2. \\ (6 - x)^2 + (1 - y)^2 = z^2. \\ y = 2x. \end{cases}$

3. $\begin{cases} x + y + z = 2. \\ xy = -1. \\ xyz = -2. \end{cases}$

4. $\begin{cases} x + y + z = 20. \\ xy = 35. \\ yz = 56. \end{cases}$

5. $\begin{cases} x^2 + y^2 + z^2 = 30. \\ x + y + z = 8. \\ yz = 2. \end{cases}$

6. $\begin{cases} (x - 1)^2 + (y + 2)^2 = z^2. \\ 3x + y = 0. \\ x + 3y - z = 0. \end{cases}$

87. Problems Producing Simultaneous Equations of Two or More Unknown Quantities.

1. The sum of two numbers is 8, and the sum of their cubes is 152. What are the numbers?

2. The sum of two numbers is 280, and the sum of their cube roots is 10. What are the numbers?

3. Four cows and 5 sheep can be bought for \$220; but for \$480 twenty-eight more sheep than cows can be bought. What is the price of each?

4. The volumes of two cubes differ by 1413 cubic inches, and their edges differ by 3 inches. Find the edge of each.

5. Find two numbers whose sum, product, and difference of their squares are equal.

6. A man has a rectangular house which occupies 1200 sq. ft. in the middle of a rectangular lot 8000 sq. ft. in area. The lot extends 30 ft. beyond the house at each end and 25 ft. at each side. What are the dimensions of the lot?

7. The fore wheel of a carriage makes 88 more revolutions than the hind wheel in going a mile; but, if the circumference of each wheel is decreased two feet, the fore

wheel will make 132 more revolutions than the hind wheel in going the same distance. What is the circumference of each wheel?

8. At simple interest \$200 amounted to \$260 in a certain time, at a certain rate. If the time had been one year less, and the rate two per cent more, it would have amounted to \$264. What were the time and rate?

9. \$1500 is invested in two parts and at different rates of interest, so as to give the same income. If the first part were invested at the second rate, the income would be \$64; but if the second part were invested at the first rate, the income would be \$49. What are the rates of interest?

10. A man sold 7 cows and 8 sheep for \$320, and found that he had sold one cow less for \$200 than sheep for \$30. What was the price of each?

11. If the length of a rectangle be increased by 9 and the breadth be decreased by 2, the area is unaltered; but if the length be decreased by 3 and the breadth by 1, the area is halved. Find the sides of the rectangle.

12. If a number of two digits is divided by the sum of its digits, the quotient is 2 and the remainder 2. If it is multiplied by the sum of its digits, the product is 112. Find the number.

13. A vessel sails 110 miles with the current and 70 miles against the current in a total time of 10 hours. On a second trip it sails 88 miles with and 84 miles against the same current in the same time. Find the velocity of the current and that of the vessel in still water.

14. Two trains start from two stations at different rates of speed, each train going toward the other station. When they meet one train has travelled 108 miles more than the other, and they finish their trips in 9 and 16 hours respectively. Find the rate per hour of each train, and the distance between the stations.

15. A certain principal at a certain rate amounts to \$1248 in one year at simple interest. Were the principal \$100 greater and the rate $1\frac{1}{2}$ times as great, the amount at the end of 2 years would be \$1456. What is the principal and what is the rate?

16. A bought two pieces of cloth, which together measured 36 yards. Each piece cost as many dollars per yard as there were yards in the piece, and the cost of the first was 4 times the cost of the second piece. Find the number of yards in each piece.

17. A certain number of persons paid a bill. If there had been 10 persons more, each would have paid \$2 less; but if there had been 5 persons fewer, each would have paid \$2.50 more. Find the number of persons and the amount of the bill.

18. A man buys a certain number of railway shares for \$912. Had he bought when each share cost \$19 less, he might have purchased 4 more shares for the same money. How many shares did he buy?

88. Review Exercises in Quadratics.

A

1. Solve $\dfrac{5}{x-2} - \dfrac{4}{x} = \dfrac{3}{x+6}$.

2. Solve $x^2 + ax(1 + 3b) + 3a^2b = 0$.

3. Solve $(x^2 + x)^2 + (x^2 + x) = 42$.

4. Solve $3x - \sqrt{2x^2 + 6x + 1} = 1 - x^2$.

5. Solve $\begin{cases} x^{\frac{1}{3}} + y^{\frac{1}{3}} = 6. \\ x + y = 72. \end{cases}$

6. A number consists of three digits whose sum is 14. The square of the middle digit is equal to the product of the two end digits; and, if 594 is added to the number, the order of the digits will be inverted. What is the number?

B

1. Solve $\frac{2x}{x-3} - \frac{x-5}{2-x} = \frac{17}{4}$.

2. Solve $ax^2 + bx(1 - a^2) = ab^2$.

3. Solve $x^3 - 7x^{\frac{3}{2}} = 8$.

4. Solve $x^2 + 4x + 2\sqrt{x^2 + 4x + 11} = 13$.

5. Solve $\begin{cases} x + \sqrt{x + y} = 12 - y. \\ x^2 + y^2 = 41. \end{cases}$

6. Eight hundred persons are seated on benches of equal length. If there were 20 benches less, it would be necessary that two persons more should sit on each bench in order that all might be seated. Required the number of benches.

C

1. Solve $\frac{4}{x-3} - \frac{3}{x+5} = \frac{17}{10}$.

2. Solve $x^{\frac{2}{3}} + 5x^{\frac{1}{3}} = 14$.

3. Solve $x + \sqrt{x + 4} = 8$.

4. Solve $(a - x)^2 - (a - x)(b - x) + (x - b)^2 = (a - b)^2$

5. Solve $\begin{cases} x^3 + y^3 = 133. \\ x + y = 7. \end{cases}$

6. Form the equation whose roots shall be $-1 \pm \sqrt{2}$.

XVII. THEORY OF QUADRATIC EQUATIONS

89. Give by inspection the sum and product of the roots of the following:

1. $x^2 + 8x - 9 = 0$.
2. $x^2 - 6x - 27 = 0$.
3. $x^2 - 5x = -6$.
4. $3x^2 + 7x = 6$.
5. $5x^2 - 6x = -1$.
6. $5x - 6x^2 + 1 = 0$.
7. $17x + 2 = -8x^2$.
8. $ax^2 + bx + c = 0$.
9. $a^2x^2 - x + 2a = 0$.
10. $2bx^2 - 3ax + bx - b^2 = 0$.
11. $2ax + b^2 = x^2 + 2ab$.
12. $3x^2 - 2ax - bx = 0$.
13. $a^2x^2 - a^2 + b^2 = 0$.

Form the equations whose roots are:

14. 5, 3.
15. $-3, -4$.
16. $3, -2$.
17. $-.5, -.1$.
18. $\frac{4}{5}, \frac{3}{7}$.
19. $a - b, a + b$.
20. $a - b, b - a$.
21. $2 \pm \sqrt{3}$.
22. $7 \pm 2\sqrt{5}$.
23. $\frac{-5 \pm 2\sqrt{3}}{2}$.
24. $1 \pm \sqrt{-1}$.
25. $-2 \pm \sqrt{-2}$.
26. $-a \pm \sqrt{-b}$.
27. $\frac{-n \pm \sqrt{n^2 - 4mq}}{2m}$.

Without solving the equations, give the nature of the roots of:

28. $x^2 + 2x - 3 = 0$.
29. $x^2 + 2x + 3 = 0$.
30. $x^2 + 2x - 2 = 0$.
31. $4x^2 - 4x + 1 = 0$.
32. $3x^2 - 2x = 0$.
33. $2x^2 - 3x + 5 = 0$.
34. $2x^2 - 5x + 3 = 0$.
35. $3x^2 + 5x - 1 = 0$.
36. $-2x^2 + 3 = 7x$.
37. $5x^2 = 3$.

Find the values of a for which the roots are equal in the following:

38. $2x^2 + 4x + a = 0$. **41.** $x^2 - 2ax + 6x + 4a = 0$.

39. $ax^2 - 4x + 3 = 0$. **42.** $ax^2 - (8 + a)x + 9 = 0$.

40. $x^2+(a-3)x+a=0$. **43.** $(8a+5)x^2 = 1+12ax$.

44. In examples **38–43** above, determine for what values of a the roots will be real and unequal; and for what values they will be imaginary.

45. If a and b represent the roots of $2x^2 - 5x + 3 = 0$, find, without solving the equation, the values of: $a + b$, ab, $(a + b)^2$, $(a - b)^2$, $a^2 - b^2$, $\frac{1}{a} + \frac{1}{b}$.

46. Find the value of a such that one of the roots $100x^2 + 60x + a = 0$ may be double the other.

47. For what value of a will one root be twice the other in $a^2x^2 - ax - x^2 + 1 = 0$.

48. For what values of k does the equation $kx(x - 1) + k(k - 1) = x(k - 1)(x - 1)$ have (a) one root equal to zero; (b) equal roots.

49. $y^2 = 12x$, $y = 3x + b$. Find b so that when these equations are solved the two values of x shall be equal.

50. State what you can concerning the roots of $ax^2 + bx + c = 0$; (a) when $b = 0$; (b) when $c = 0$; (c) when a and c have opposite signs.

51. Find m so that one root of $x^2 + mx + 24 = 0$ shall exceed the other by 2.

52. Find, without solving the equation, the remaining root in each of the following:

(a) $x^2 - 20x + 96 = 0$, one root being 12.
(b) $25x^2 - 20x + 3 = 0$, one root being $\frac{1}{5}$.
(c) $x^2 - 4x + 1 = 0$, one root being $2 + \sqrt{3}$.
(d) $x^2 - 2x + 3 = 0$, one root being $1 - \sqrt{-2}$.

Factor the following:

53. $x^2 - 20x - 341$.

54. $x^2 - 2x - 2$.

55. $x^2 - 2x - 1$.

56. $5x^2 + 14x - 55$.

57. $4x^2 - 12x + 7$.

58. $2x^2 + 3xy + x + y^2 - y - 6$.

XVIII. GRAPHS

90. Locate graphically the following points, taking the first number of each pair as the abscissa, the second as the ordinate:

1. (4, 3); (−3, −5); (5, −6); (−4, 5); (0, 4); (7, 0); (−5, 0); (0, 0).

2. (3, $-2\frac{1}{2}$); ($-3\frac{1}{4}$, 2); (−5, $-3\frac{1}{2}$); (0, $-6\frac{1}{2}$); (4, $2\frac{1}{4}$); (3, −4.5); (−3.9)(−7.1).

3. (1, $\sqrt{5}$); (1, $-\sqrt{5}$); (0, 2); (0, −2); (2, 0); (−2, 0); (−1, $\sqrt{5}$); (−1, $-\sqrt{5}$); (2, $\sqrt{8}$); (2, $-\sqrt{8}$); (−2, $\sqrt{8}$); (−2, $-\sqrt{8}$).

4. Construct the triangle whose vertices are at the points (−3, −1), (−1, 3), (3, −3).

5. Draw the straight line on which the following points are situated: (6, −1); (−6, −5); (12, 1). Give the coordinates of other points on the same line.

Draw the graphs of the following equations:

6. $y = x + 4$.

7. $x - 2y = 5$.

8. $3x - 2y = 3$.

9. $x = -y$.

10. $4y = x$.

11. $6y - 5x = -2$.

12. $2x - 3y = -2$.

13. $2x - 3y = -6$.

14. $4x + 3y = 0$.

15. $2x - 4y = -12$.

Solve graphically each pair of the following simultaneous equations:

16. $\begin{cases} 2x - 3y = -1. \\ x - y = 1. \end{cases}$

17. $\begin{cases} 2x - 3y = -1. \\ x + 4y = 16. \end{cases}$

18. $\begin{cases} x + y = 0. \\ x = 2y. \end{cases}$ **21.** $\begin{cases} 5x - 4y = -22. \\ x + 2y = 4. \end{cases}$

19. $\begin{cases} y = 3x - 6. \\ y = 0. \end{cases}$ **22.** $\begin{cases} 2x + 3y = 9. \\ 3x + y = -4. \end{cases}$

20. $\begin{cases} x + 5y = 3. \\ 4x - 2y = 1. \end{cases}$ **23.** $\begin{cases} 2x - 3y = 14. \\ 3x - y = -7. \end{cases}$

24. Show graphically why the equations $\begin{cases} 2x - 3y = 4 \\ 8x - 12y = 12 \end{cases}$ cannot be solved simultaneously.

91.

Graph each of the following equations, and estimate from the graph the roots of the equation formed by making y equal to zero:

1. $y = x^2 - 5x + 6.$
2. $y = x^2 - 4x - 5.$
3. $y = x^2 - 6x + 9.$
4. $y = x^2 - 5x + 5.$
5. $y = x^2 - 1.$
6. $y = -x^2 - 2x + 8.$
7. $y = 2x^2 - 3x - 10.$
8. $y = 4x^2 + 4x + 1.$
9. $y = -2x^2 + 11x - 5$
10. $y = x^2 - 2x - 2.$

Graph the following:

11. $y = x^2 - 3x + 5.$
12. $x^2 + y^2 = 25.$
13. $x^2 - y^2 = 9.$
14. $y = x^2 - 4x.$
15. $xy = 4.$
16. $9x^2 + 4y^2 = 36.$
17. $x^2 + y^2 = 17.$
18. $4x^2 - xy = 6.$
19. $x^2 - xy + y^2 = 25.$
20. $x^2 + 2y^2 - 2x = 15.$

Graph the following systems and from the graphs estimate the roots for each pair:

21. $\begin{cases} x^2 = 4y. \\ x + 2y = 4. \end{cases}$ **23.** $\begin{cases} x^2 + y^2 = 40. \\ x + 2y = 10. \end{cases}$

22. $\begin{cases} x^2 + 9y^2 = 81. \\ x - 3y = 5. \end{cases}$ **24.** $\begin{cases} xy = 3. \\ x + y = 4. \end{cases}$

25. $\begin{cases} x^2 + y^2 = 25. \\ xy = 12. \end{cases}$

26. $\begin{cases} x^2 + y^2 = 6y. \\ x + 6 = y. \end{cases}$

27. $\begin{cases} x^2 - y^2 = 9. \\ y^2 - x = 6. \end{cases}$

28. $\begin{cases} xy = 6. \\ 3x - 2y = 5. \end{cases}$

29. $\begin{cases} 4x^2 - xy = 6. \\ 3xy - y^2 = 5. \end{cases}$

30. $\begin{cases} y^2 - x^2 = 16. \\ 2x^2 + y^2 = 64. \end{cases}$

XIX. RATIO AND PROPORTION

92. Exercises in Ratio.

1. Arrange in order of magnitude the ratios 3 : 4, 5 : 7, 11 : 14.

Find the ratio of x to y in the five following:

2. $x = 2y$

3. $8x = 3y$.

4. $ax + by = cx - dy$.

5. $15(2x^2 - y^2) = 7xy$.

6. $6(x^2 + y^2) = 13xy$.

7. If $x : y = 3 : 4$, find the ratio of $7x - 4y : 3x + y$.

8. If $\frac{y}{x - z} = \frac{y + x}{z} = \frac{x}{y}$, find the ratios of $x : y$ and $y : z$.

9. If 14 cows and 12 sheep are worth as much as 5 cows and 48 sheep, what is the ratio of the value of a cow to that of a sheep?

93. Exercises in Proportion.

Find a fourth proportional to:

1. 3, 5, and 7.

2. $a^3 + y^3$, $x^2 - xy + y^2$, and $x + y$.

Find a mean proportional between:

3. a^3b and ab^3.

4. $(a + b)^2$ and $(a - b)^2$.

Find a third proportional to:

5. $(a-b)^2$ and a^2-b^2.

6. $\frac{x}{y} + \frac{y}{x}$ and $\frac{x}{y}$.

If $a : b = c : d$, prove:

7. $a + b : a - b = c + d : c - d$.

8. $a : b = \sqrt{a^2 + c^2} : \sqrt{b^2 + d^2}$.

9. $\sqrt{a^2 + b^2} : \sqrt{c^2 + d^2} = \sqrt[3]{a^3 + b^3} : \sqrt[3]{c^3 + d^3}$.

10. $a + b : c + d = \sqrt{2a^2 - 3b^2} : \sqrt{2c^2 - 3d^2}$.

11. $a^2 + ab + b^2 : c^2 + cd + d^2 = a^2 - ab + b^2 : c^2 - cd + d^2$.

12. $a^2 + 3ab + b^2 : c^2 + 3cd + d^2 = 2ab + 3b^2 : 2cd + 3d^2$.

If $a : b = b : c$, prove:

13. $a + b : b + c = \sqrt{a} : \sqrt{c}$.

14. $a^2 + ab + b^2 : b^2 + bc + c^2 = a : c$.

15. $a^3 + c^3 : a^2 - b^2 + c^2 = a + c : 1$.

If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, prove:

16. $\frac{x + y}{a + b} = \frac{y + z}{b + c} = \frac{z + x}{c + a}$.

17. $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax + by + cz)^2$.

From each of the following prove $a : b = c : d$.

18. $(a + b - 3c - 3d)(2a - 2b - c + d) = (2a + 2b - c - d)(a - b - 3c + 3d)$.

19. $(3a + 6b + c + 2d)(3a - 6b - c + 2d) = (3a - 6b + c - 2d)(3a + 6b - c - 2d)$.

Solve the following equations, using theorems in proportion:

20. $3 + \sqrt{x} : 3 - 2\sqrt{x} = 11 : 5$

21. $\frac{x + \sqrt{12a - x}}{x - \sqrt{12a - x}} = \frac{\sqrt{a} + 1}{\sqrt{a} - 1}$.

22. $\frac{3x^4 + x^2 - 2x - 3}{3x^4 - x^2 + 2x + 3} = \frac{5x^4 + 2x^2 - 7x + 3}{5x^4 - 2x^2 + 7x - 3}$.

23. $\dfrac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}} = \dfrac{m+n}{m-n}.$

24. $\dfrac{\sqrt{2a^2-x^2}+b\sqrt{2a-x}}{\sqrt{2a^2-x^2}-b\sqrt{2a-x}} = \dfrac{\sqrt{a}+b}{\sqrt{a}-b}.$

25. $\begin{cases} x^3+y^3 : x^3-y^3 = 70 : 38. \\ xy = 6. \end{cases}$

94. Problems in Ratio and Proportion.

1. An express train goes from A to B in $2\frac{1}{4}$ hours; a horse goes the same distance in 9 hours. What is the ratio of their rates?

2. Divide \$1000 between A and B so that A receives \$7 to every \$1 B receives.

3. Find two numbers such that their product, their sum, and three times their difference are to each other as 5 : 2 : 4.

4. The sum of two numbers in the ratio 2 : 5 is 24. Find the numbers.

5. The difference of the squares of two numbers in the ratio 3 : 2 is 45. Find the numbers.

6. If $3+2\sqrt{x} : 3-2\sqrt{x} = 11 : 5$, find x.

7. From the relation $x+y : x-y = 7 : 3$, determine the value of $x : y$.

8. Find two numbers such that if 9 is added to each they are in the ratio 5 : 6; and if 6 is subtracted from each they are in the ratio 2 : 3.

9. If gunmetal consists of 9 parts of copper to 1 part of tin, how many pounds of each are there in 20 pounds of gunmetal?

10. A mean proportional between two numbers is $2\sqrt{14}$ and the sum of their squares is 113. Find the numbers.

11. What number must be added to each of the terms of the ratio $m : n$ that it may become equal to the ratio $p : q$?

12. The ratio of one number to another is equal to that of p to q. If m be added to the first number and n to the second, the ratio of the numbers thus formed is equal to that of r to s. Find the numbers.

13. A line 16 inches long is divided into two parts such that the longer part is a mean proportional between the whole line and the shorter part. Find the length of the shorter part to two places of decimals.

14. Four numbers are in proportion; the difference between the first and the third is $2\frac{2}{3}$; the sum of the second and third is $6\frac{1}{3}$; and the third is to the fourth as 4 : 5. Find the numbers.

15. A and B do a piece of work for \$60. A works 6 days of 8 hours each and B works 8 days of 10 hours each. How much of the \$60 should each receive?

XX. VARIATION

95. Exercises in Variation.

1. Express in symbolic form each of the following: x varies as y; x varies inversely as y; x varies jointly as y and z; x varies directly as y and inversely as z.

2. Express each form in Exercise 1 as an equation.

3. If $x \propto y$, and equals 8 when $y = 15$, what is its value when $y = 10$?

4. If $x \propto \frac{1}{y}$, and equals 15 when $y = 4$, what is its value when $y = 10$?

5. If $x \propto yz$, and $y = 5$ when $x = 9$ and $z = 7$, what is the value of y when $z = 70$ and $x = 54$?

6. If $x \propto \frac{y}{z}$, and $y = 10$ when $x = 14$ and $z = 14$, what is the value of z when $x = 49$ and $y = 45$?

7. If $x \propto y$, prove that $x^n \propto y^n$; also that $x^2 + y^2 \propto x^2 - y^2$.

8. If $x \propto y$ and $y \propto z$, prove that $x \pm y$ and $\sqrt{xy}$ each vary as z.

9. If $A \propto BC$, $B \propto D^2$, and $C \propto \frac{1}{A}$, prove that A varies as D.

10. If x varies directly as $\sqrt{y}$ and inversely as z^3, and $x = 3$ when $y = 256$ and $z = 2$, what is the value of y when $x = 24$ and $z = \frac{1}{2}$?

11. x varies as the sum of two quantities, one of which is constant and the other varies as xy; and when $x = 3$, $y = 1$; also when $x = 9$, $y = 4$. Find the value of x in terms of y.

12. x varies as the difference of two quantities, one of which is constant and the other varies as y^2; and when $x = 3$, $y = 2$; also when $x = 18$, $y = 1$. Find the value of x in terms of y.

13. x varies as the sum of two quantities, one of which varies as y and the other as y^2. If $x = 7$ when $y = 1$, and $= 39$ when $y = 3$, what is the value of x in terms of y?

14. How many acres can 16 men reap in 3 days, if 7 men can reap 126 acres in 20 days?

15. How many bushels of grain will 30 horses eat in 16 days, if 12 horses eat 48 bushels in 20 days?

16. The area of a circle whose radius is 10 feet is 314.159 square feet. Find the area of a circle whose radius is 12 feet, from the law that the area of a circle varies as the square of its radius.

17. From the same law show that the area of a circle $2\frac{1}{2}$ inches in diameter is equal to the sum of the areas of two circles $1\frac{1}{2}$ and 2 inches in diameter.

18. If the volume of a sphere varies as the cube of its radius, find the radius of a sphere whose volume equals that of the sum of two spheres whose radii are respectively 6 feet and 3.5 feet.

19. The illumination from a source of light varies inversely as the square of the distance. How far must a screen that is 10 feet from a lantern be moved in order to receive one fourth as much light?

XXI. ARITHMETICAL PROGRESSION

96.

In each series give the value of a, d, and n. Find l and S.

1. 7, 11, 15, ... to 13 terms.

2. 5, 8, 11,. . . . . to 12 terms.

3. 63, 58, 53, ... to 8 terms.

4. $3x, 5x, 7x, \ldots$ to 36 terms.

5. $4, 5\frac{1}{4}, 6\frac{1}{2}, \ldots$ to 13 terms.

6. $1, -\frac{1}{6}, -\frac{7}{6}, \ldots$ to 11 terms.

7. .2, 1, 1.8, ... to 12 terms.

8. $a+b, a, a-b, \ldots$ to 12 terms.

9. $3a-\frac{1}{a}, 2a, a+\frac{1}{a}, \ldots$ to 12 terms.

10. 2, 5, 8, ... to 2 r terms.

11. Find the sum of the first n positive even numbers.

Which term of the series:

12. 5, 8, 11, . . . is 65?

13. 4, 11, 18, . . . is 312?

14. $\frac{7}{6}, \frac{4}{3}, \frac{8}{2}, \ldots$ is 18?

How many terms must be added of the series:

15. 42, 39, 36, . . . to make the sum 315?

16. $-16, -15, -14, \ldots$ to make the sum 74?

17. $-10\frac{1}{2}, -9, -7\frac{1}{2}, \ldots$ to make the sum -42?

Find the first three terms:

18. When the 27th term is 186, and the 45th 312.

19. When the 15th term is 25, and the 29th 46.

20. When the 16th term is 214, and the 51st 739.

21. Insert 8 arithmetical means between 1 and 0.

22. Insert 22 arithmetical means between 8 and 54.

23. Find the arithmetical mean between $7\frac{3}{4}$ and $-23\frac{1}{4}$.

24. Find the arithmetical mean between a and

$$\frac{an + 2b - a}{n + 1}.$$

25. Given $a = 7$, $d = 2$, $S = 1927$; find l and n.

26. Given $a = 10$, $n = 5$, $S = -20$; find l and d.

27. Given $d = \frac{3}{4}$, $l = 41$, $S = 1127$; find n and a.

28. The forms x, $x + y$, $x + 2y$, etc.; $x - 3y$, $x - y$, $x + y$, $x + 3y$; $x - 2y$, $x - y$, x, $x + y$, $x + 2y$, are sometimes used to represent the terms of a series. What is the common difference in each, and when would one form be preferable to another?

29. The sum of three numbers in A.P. is 27, and their product is 504. What are the numbers?

30. The sum of four numbers in A.P. is 20, and the sum of their squares is 120. What are the numbers?

31. The sum of five numbers in A.P. is 75, and the product of the greatest and least is 161. What are the numbers?

32. The sum of three numbers in A.P. is 12, and the sum of their squares is 66. What are the numbers?

33. A sets out from a place and travels uniformly $2\frac{1}{2}$ miles per hour. B sets out from the same place 3 hours afterwards and follows A at the rate of 3 miles the first hour, increasing his rate one half a mile each hour. In how many hours will he overtake A?

34. A man agrees to pay a debt of \$3600 in 40 annual payments which form an Arithmetical Progression. When

30 of these payments are made he dies, leaving one-third of the debt unpaid. What was the first payment?

35. Sum the series 1, 3, 5, 7, . . . to $2n$ terms.

36. The sum of n terms of a series is n^2. Find the first term and common difference, the last term being $2n - 1$.

Find the sum of n terms of the series:

37. $\frac{2}{3}, \frac{3}{4}, \ldots$

38. $(a - b), (3a - 2b), \ldots$

39. $\frac{a-1}{a}, \frac{a-2}{a}, \frac{a-3}{a}, \ldots$

40. Find the sum of all integers between 100 and 500 which are divisible by 6.

41. Find the sum of $n + 2$ terms of the series

$$a, \frac{na + c}{n + 1}, \frac{(n-1)a + 2c}{n + 1}, \ldots$$

42. If a^2, b^2, c^2 are in A.P., prove that

$$\frac{1}{b + c}, \frac{1}{c + a}, \frac{1}{a + b} \text{ are in A.P.}$$

97. Derivation of Formulas.

The formulas on the following page, selected from a possible twenty, are to be derived from the fundamental formulas

$$l = a + (n - 1)d, \text{ and } S = \left(\frac{a + l}{2}\right)n.$$

They have no especial value in solving problems and need not be committed to memory. Their derivation is introduced as an excellent exercise in the reduction of literal equations; in the changing of forms, given or obtained, to required forms; and in using any letter as the unknown quantity.

No.	Given	Required	Formulas
1.	dnS	l	$l = \frac{S}{n} + \frac{(n-1)d}{2}$.
2.	dnl	S	$S = \frac{1}{2}n[2l - (n-1)d]$.
3.	adl	S	$S = \frac{l+a}{2} + \frac{l^2 - a^2}{2d}$.
4.	anS	d	$d = \frac{2(S - an)}{n(n-1)}$.
5.	alS	d	$d = \frac{l^2 - a^2}{2S - l - a}$.
6.	nlS	d	$d = \frac{2(nl - S)}{n(n-1)}$.
7.	adS	l	$l = -\frac{1}{2}d \pm \sqrt{2dS + (a - \frac{1}{2}d)^2}$.
8.	dlS	a	$a = \frac{1}{2}d \pm \sqrt{(l + \frac{1}{2}d)^2 - 2dS}$.
9.	adS	n	$n = \frac{d - 2a \pm \sqrt{(d-2a)^2 + 8dS}}{2d}$.
10.	dlS	n	$n = \frac{2l + d \pm \sqrt{(2l+d)^2 - 8dS}}{2d}$.

XXII. GEOMETRICAL PROGRESSION

98. Find the required terms in each of the following:

1. The 5th and 8th in the series 3, 6, 12, . . .

2. The 7th and 14th in the series $\frac{1}{64}$, $\frac{1}{32}$, $\frac{1}{16}$, . . .

3. The 6th and 9th in the series .008, .04, .2, . . .

4. The 50th in the series $2x$, 1, $\frac{1}{2x}$, . . .

5. The nth term in the series x, x^4, x^7, . . .

6. The $2n$th term in the series 3, -3^2, 3^3, . . .

Find the sum of:

7. Five terms of the series 9, 3, 1, . . .

8. Eight terms of the series $1\frac{1}{2}$, $1\frac{1}{3}$, $1\frac{5}{27}$, . . .

9. n terms of the series 8, 12, 18, . . .

10. Six terms of the series 1, 1.5, 2.25, . . .

11. Nine terms of the series $-\frac{8}{9}$, $1\frac{1}{3}$, -2, . . .

12. Eight terms of the series $\frac{1}{\sqrt{3}}$, 1, $\sqrt{3}$, . . .

13. Five terms of the series $\sqrt{2} - 1$, 1, $\sqrt{2} + 1$, . . .

Find n in each of the series when:

14. $l = 1024$, $a = 8$, $S = 2040$.

15. $l = \frac{1}{2\,4\,3}$, $a = 3$, $r = -\frac{1}{3}$.

16. $S = \frac{8\,8\,9}{1\,6\,6}$, $a = \frac{2}{6}$, $r = \frac{5}{4}$.

Find the required letters in the following:

17. Given $a = 10$, $r = 2$, $S = 2550$; find n and l.

18. Given $l = 243\sqrt{3}$, $r = \sqrt{3}$, $S = 364(\sqrt{3} + 1)$; find a and n.

19. Given $a = 2$, $n = 10$, $l = 1024$; find r and S.

20. Given $a = \frac{2}{3}$, $r = -\sqrt{3}$, $n = 8$; find S and l.

21. Given $a = \frac{1}{\sqrt{2}}$, $l = \frac{512}{\sqrt{2}}$, $S = \frac{1023}{\sqrt{2}}$; find n and r.

Find the sum to infinity of:

22. $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \ldots$ **23.** $\frac{4}{5}, 1, \frac{5}{4}, \ldots$

24. $1, -\frac{1}{2}, \frac{1}{4}, \ldots$

25. $\frac{\sqrt{2}+1}{\sqrt{2}-1}, \frac{1}{2-\sqrt{2}}, \frac{1}{2}, \ldots$

26. $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}+1}, \frac{2}{4+3\sqrt{2}}, \ldots$

27. .151515 . . . **29.** .42323 . . .

28. .281313 . . . **30.** .123123 . . .

31. 32.020352035 . . .

32. Find the geometrical mean between 12 and 108.

33. Find the geometrical mean between $4x^2 - 12x + 9$ and $9x^2 + 12x + 4$.

34. Insert five geometrical means between 1 and 64.

35. Insert 2 geometrical means between 3 and 24.

36. Insert 3 geometrical means between 2 and 32.

37. Insert 7 geometrical means between 81 and $\frac{256}{81}$.

38. Insert 6 geometrical means between $\frac{a}{\sqrt{3}}$ and $27a$.

39. Find the geometrical mean of $x^{n-1}y^{2n}$ and $x^{n+1}y^2$.

40. If the arithmetical mean between two numbers is $1 + a^2$ and the geometrical mean $1 - a^2$, what are the numbers?

41. If a, b, c, and d are in Geometrical Progression, prove that $(b-c)^2 + (c-a)^2 + (d-b)^2 = (a-d)^2$.

42. x, xy, xy^2, etc.; $\frac{x}{y}, x, xy$; $\frac{x^2}{y}, x, y$, $\frac{y^2}{x}$, are forms used in solving questions in Geometrical Progression. What is the ratio in each, and when would one be preferred to another?

43. The sum of three terms in Geometrical Progression is 63, and the third is 45 greater than the first. What are the terms?

44. The sum of the first 8 terms of a Geometrical Progression is 3280, and the sum of the first 4 terms is 40. What are the first three?

45. The sum of three terms of a Geometrical Progression is 21, and the sum of their squares is 189. What are the terms?

46. The continued product of four numbers in Geometrical Progression is 2916, and the sum of the means is 15. What are the numbers?

99. Derivation of Formulas.

From the formulas $l = ar^{n-1}$ and $S = \dfrac{lr - a}{r - 1}$, derive each of the following formulas:

No.	Given	Required	Formulas
1.	arS	l	$l = \dfrac{a + (r-1)S}{r}$.
2.	rnS	l	$l = \dfrac{(r-1)Sr^{n-1}}{r^n - 1}$.
3.	arn	S	$S = \dfrac{a(r^n - 1)}{r - 1}$.
4.	anl	S	$S = \dfrac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}$.
5.	rnl	S	$S = \dfrac{(r^n - 1)l}{(r-1)r^{n-1}}$.
6.	rnS	a	$a = \dfrac{(r-1)S}{r^n - 1}$.
7.	rlS	a	$a = rl - (r-1)S$.
8.	alS	r	$r = \dfrac{S - a}{S - l}$.

100. Miscellaneous Problems in Progressions.

From College Entrance Examinations

1. Find the last term and the sum of the terms of the progression $-2, 6, -18 \ldots$ to eight terms.

2. Determine whether the numbers $3\frac{3}{5}$, $4\frac{4}{5}$ and $6\frac{2}{3}$ are in arithmetic or geometric progression and find the sum of the first six terms by the general formula.

3. The second term of a geometrical progression is $3\sqrt{2}$ and the fifth term is $\frac{3}{16}$. Find the first term and the ratio.

4. A body falling freely from rest falls 16 feet the first second, 48 feet the second second, 80 feet the third second, and so on. Find the distance fallen during the seventh second, and the distance fallen during 7 seconds from rest.

5. The successive swings of a pendulum in a resisting medium form a geometrical progression. If the first swing is 2 cm., and the second 1.9 cm., what will be the length of the sixth swing?

6. Find the twenty-first term of the series $\frac{1}{\sqrt{2}}, \sqrt{2}, \frac{3\sqrt{2}}{2}, \ldots$

7. The arithmetic mean between two numbers is 39 and the geometric mean is 15. Find the numbers.

8. The difference between two numbers is 48. The arithmetic mean exceeds the geometric mean by 18. Find the numbers.

9. If the population of a town that now has 2665 inhabitants were doubled every ten years, how many people would the town contain at the end of a century?

10. A ball starting from rest rolls down an inclined plane, passing over 5 inches in the first second, 15 inches in the

second second, 25 inches in the third second, etc. How long will it take the ball to pass over a distance of 15 feet?

11. Each stroke of an air pump removes $\frac{1}{15}$th of the air then in the receiver. How much of the air originally in the receiver is removed in 10 strokes?

12. The sum of an arithmetical progression of n terms is $4n^2$, the common difference is twice the first term. What is the progression?

13. The sum of three numbers in geometrical progression is 70. If the first be multiplied by 4, the second by 5, and the third by 4, the resulting numbers will be in arithmetical progression. Find the three numbers.

14. A number consists of three digits which are in arithmetical progression. When the number is divided by the sum of its digits, the quotient is 26. If 198 is added to the number, the order of the digits is reversed. Find the number.

15. There are four numbers of which the first three are in arithmetical progression, and the first, the second, and the fourth are in geometrical progression. The sum of the first and the third is 8. The sum of the second and the fourth is 36. Find the numbers.

16. There are m arithmetic means between 1 and 31, and the seventh mean is to the $(m - 1)$st mean as 5 is to 9. What is the number of means?

17. The sum of three numbers in geometric progression is $12\frac{2}{3}$. If the first number is multiplied by $\frac{1}{9}$, the second by $\frac{1}{10}$, and the third by $\frac{2}{3}$, the results are in arithmetic progression. Find the numbers.

18. The sum of an infinite geometric series is 4 and the first term is 6. Find the ratio and the sum of 4 terms.

19. Insert between 2 and 9 two numbers, such that the

first three of the four may be in arithmetical progression, and the last three in geometrical progression.

20. An elastic ball bounces to three fourths the height from which it falls. If it is thrown up from the ground to a height of 15 feet, find the total distance traveled before it comes to rest.

21. Three numbers, the middle one of which is -1, form an arithmetical progression; if the smallest number and the middle number are each increased by 4, and the largest number is decreased by 12, the results form a geometrical progression. Find the two unknown numbers.

22. A man was paid for boring a well at the rate of 3.24 marks for the first meter, 3.29 marks for the second, 3.34 marks for the third, and so on. The well had to be sunk 100 meters. How much was paid for boring the well?

23. The sum of the $2r$ terms of the series 1, 3, 5, . . ., $(2r - 1)$ is to the sum of the last r terms as $x : 1$. Find the value of x.

24. A person contributes one cent and sends letters to three friends asking each to contribute one cent to a certain charity and to write a similar letter to each of three friends, each of whom is to write three letters and so on until ten sets of letters have been written. If all respond, how much money will the charity receive?

25. Twelve potatoes are placed in line at distances 6, 12, 18 . . . feet from a basket. A boy starting from the basket picks up the potatoes and carries them one at a time to the basket. How far must he run to complete the potato race?

26. B travels 3 miles the first day, 7 miles the second day, 11 miles the third day, etc.; in how many days will B overtake A who started from the same point 8 days in advance and who travels uniformly 15 miles a day?

27. The first term of an arithmetical progression is 2, and

the first, second, and fifth terms are in geometrical progression. Find the sum of 11 terms of the arithmetical progression.

28. Insert between 6 and 16 two numbers, so that the first three of the series may be in arithmetic progression and the last three in geometric progression.

29. The sum of an infinite number of terms of a geometric progression is 4, and the sum of the cubes of the terms is 192; find the first term and the ratio.

30. On Jan. 1, 1911 a man has $500. His annual income is $1200, and his living expenses for the year 1911 will be $800. If his income remains the same, but the cost of living increases $20 per year, in how many years will he have $3690?

31. Insert between 1 and 21 a series of arithmetic means such that the sum of the last three shall be equal to 48.

XXIII. THE BINOMIAL THEOREM

101. Expand the following:

1. $(a + b)^4$.
2. $(a - b)^5$.
3. $(x - y)^6$.
4. $(a^2 + b^2)^4$.
5. $(a^3 - b^2)^3$.
6. $(2a - 3b)^5$.
7. $(4a^2 + 5b^2)^4$.
8. $(1 - 3x)^5$.
9. $(x - 5x^2)^5$.
10. $\left(\frac{a}{2} - \frac{b}{3}\right)^4$.
11. $\left(2a - \frac{b}{2}\right)^6$.
12. $\left(\frac{x^2}{3} - 2a^{-1}\right)^5$.
13. $(\frac{1}{2}x^{-2} + 2x^2)^4$.
14. $(a + x)^n$ to 4 terms.
15. $(a^{-3} - \frac{1}{3}x^{\frac{1}{2}})^6$.
16. $(2a + \sqrt{a})^4$.
17. $(\sqrt[3]{b^2} - 3\sqrt{b})^3$.
18. $(\frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{-2})^6$.
19. $\left(\frac{3\sqrt{b}}{\sqrt[4]{a^3}} - \frac{\sqrt[4]{a}}{b^3\sqrt{b}}\right)^4$.
20. $(x^a - x^{-2a})^6$.
21. $(x^2 + 2^{-1}x^{-\frac{1}{2}})^{12}$ to 3 terms.
22. $\left(\frac{3}{\sqrt[3]{x^2}} - \frac{x\sqrt{x}}{3}\right)^5$.
23. $(a - x)^{2n}$ to 3 terms.
24. $(a^2 - 3a + 2)^3$.
25. $(1 + x + x^2)^3$.
26. $(a - 2b + b^2)^4$.

102. To Find any Particular Term of an Expansion.

Find the

1. Fourth term of $(x - y)^{12}$.
2. Fifth term of $(x + y)^6$.
3. Fifth term of $(a + b)^{16}$.
4. Sixth term of $(a^2 - b^2)^{10}$.
5. Eighth term of $(M - 1)^{12}$.
6. Tenth term of $(x - y)^{20}$.
7. Sixth term of $(a^3 + b^3)^{11}$.

8. Fifth term of $(2a + 3b)^8$.

9. Sixth term of $(3x - y)^7$.

10. Fourth term of $(2x - \frac{1}{2}y)^{12}$.

11. Fourth term of $\left(\frac{x^2}{2} - \frac{y^2}{3}\right)^{15}$.

12. Seventh term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$.

13. Middle term of $\left(\frac{2x}{3} - \frac{3a}{2}\right)^6$.

14. Term containing x^{16} in $\left(x^2 - \frac{a}{2x}\right)^{14}$.

15. Term containing a^8 in $(1 - 3a^2)^8$.

16. Term not containing x in $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$.

17. Term not containing x in $\left(x - \frac{1}{x^2}\right)^{12}$.

18. Fifteenth term of $(a^2 - 2b^2)^{18}$.

19. Fourth term of $\left(\sqrt{a} - \frac{b^7}{6a^5}\right)^{21}$.

20. Tenth term of $\left(\frac{2a^2}{b^3} - \sqrt[5]{b^2}\right)^{19}$.

21. Middle term of $\left(\sqrt{y} - \frac{2\sqrt{x}}{y}\right)^8$.

22. Eighth term of $\left(\frac{4\sqrt[4]{b^3}}{3a^5} - \frac{1}{2}ab^{-\frac{1}{5}}\sqrt{a}\right)^9$.

23. The term of $(x^3 + 2x^{-\frac{1}{3}})^{15}$ which contains x^{15}.

24. The coefficient of x^{-1} of $\left(\frac{\sqrt{x}}{\sqrt[3]{a^2}} - \frac{a\sqrt{a}}{3x^3}\right)^{12}$.

25. The coefficient of x^{15} in $x^3(1 - x^2)^{12}$.

REVIEW

103. General Review of "Quadratics and Beyond."

Exercises from College Entrance Examinations.

1. Solve the equations (*a*) $abcx^2 - (a^2b^2 + c^2)x + abc = 0$. (*b*) $x^2 - x + 2\sqrt{x^2 - x - 11} = 14$.

2. (*a*) Solve for x the equation: $11x^2 - 19 = 66x + 58$. (*b*) Find by the graphic method two approximate values of x in the equation $x^2 - 5x - 15 = 0$.

3. A pays \$420 for the rent of a farm. He lets all but four acres to B, charging \$2.50 per acre more than he paid, and receives from B as much as he originally paid. How many acres in the farm?

4. Solve the simultaneous equations $x^2 + 4y^2 + 4x = 0$, $x + 2y + 4 = 0$. Arrange your answers in corresponding pairs and verify.

5. The three dimensions of a rectangular box are in the ratio 1:3:5, and its total outside surface is 22.54 square meters; find the length of the diagonal of the box.

6. In a geometric series of 5 terms the difference between the first and the fifth term is 160; the difference between the second and the fourth term is 48. Find each term of the series.

7. Write the fifth term of $\left(\frac{x}{y} - \sqrt{\frac{y}{x}}\right)^8$.

8. Determine for what values of a the roots of $4x^2 - 2(a - 3)x + 1 = 0$ are (1) real and equal; (2) real and unequal; (3) imaginary.

9. Determine by inspection the nature of the roots of the following equations:

(*a*) $2x^2 - 3x + 5 = 0$; (*b*) $3x^2 - 5x = 2$; (*c*) $9x^2 - 12x + 4 = 0$.

10. Solve the simultaneous equations:

$$x^2 + xy + y^2 = 19,\ x^2 - xy + y^2 = 7.$$

11. What number added to 1, 13, and 73, will give results in geometrical progression?

12. Solve for x: (*a*) $4x^{-\frac{4}{3}} - 5x^{-\frac{2}{3}} + 1 = 0$.

(*b*) $10x^{1\frac{5}{2}} = 3(x^{\frac{1}{2}} + x^{\frac{1}{3}})$.

13. Solve by completing the square the equation $ax^2 + bx + c = 0$, and from this result find tests for determining the character of the roots.

14. Find the first three terms of the expansion of

$$\left(x^2 + \frac{1}{2\sqrt{x}}\right)^{12}$$

and simplify the result. Also find the middle term.

15. Solve the equations:

(*a*) $acx^2 - bcx - adx + bd = 0$.

(*b*) $a^2x^2 - 2a^3x + a^4 - 1 = 0$.

(*c*) $\dfrac{x}{a^2b(x+c)} = \dfrac{x+c}{d^2bx}$.

(*d*) $\begin{cases} ay^2 + bxy = b. \\ bx^2 + axy = a. \end{cases}$ (find x and y.)

16. Construct the equation whose roots are

$$\frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}.$$

17. Solve the simultaneous equations $\begin{cases} x^2 + xy = 12. \\ xy - 2y^2 = 1. \end{cases}$

18. Solve $\sqrt{x^2 - a^2} = 2 - \dfrac{1}{\sqrt{x^2 - a^2}}$.

19. Find 3 numbers in the ratio of 1:2:3, such that the sum of their squares is 350.

20. Solve $9x - 3x^2 + 4\sqrt{x^2 - 3x + 5} = 11$.

21. A man deposits in the bank 1 cent the first day, 2 cents the next day, 4 cents the third day, and so on for sixteen days: find the whole amount deposited.

22. Plot $2x - 3y = 6$ and $2y + 1 = 4x - 4x^2$, using the same axes, and estimate from the graphs the solutions of the equations.

23. (*a*) In an arithmetical progression the sum of the first six terms is 261, and the sum of the first nine terms is 297. Find the common difference.

(*b*) Three numbers whose sum is 27 are in arithmetical progression. If 1 is added to the first, 3 to the second, and 11 to the third, the sums will be in geometrical progression. Find the numbers.

(*c*) Derive the formula for the sum of n terms of a geometrical progression.

24. The so-called effective area of a chimney is given by

$$E = A - \tfrac{8}{5}\sqrt{A},$$

where A is the measured area. Find A when E is 28 square feet.

25. Divide 20 into four parts which are in arithmetical progression and such that the product of the first and fourth is to the product of the second and third in the ratio of 2 : 3.

26. Find the coefficient of x^4 in $\left(2x^2 - \frac{1}{4x}\right)^8$.

27. Solve (*a*) $\dfrac{x - p + q}{x + p - q} = \dfrac{p - q - x}{p + q + x}$;

(*b*) $\dfrac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} = 3 - x.$

28. Solve for x: $\sqrt{\frac{x^2+3}{x}} - \sqrt{\frac{x}{x^2+3}} = \frac{3}{2}$.

29. Two trolley cars of equal speed leave A and B, which are 20 miles apart, at different times. Just as the cars pass each other an accident reduces the power and their speed is decreased 10 miles per hour. One car makes the journey from A to B in 56 minutes and the other from B to A in 72 minutes. What is their common speed?

30. For an isosceles triangle we have the relation

$$a = \sqrt{2R^2 + R\sqrt{4R^2 - c^2}},$$

where a is one of the equal sides, c the base, and R the radius of the circumscribed circle. Find the value of R in terms of a and c.

31. Construct the quadratic equation the product of whose roots shall equal twice the sum of the roots of the equation, $2x^2 - (a^2 - b^2)\ x - 5a^4 = 0$, and the sum of whose roots shall equal ten times the product of the roots of the equation, $10x^2 + (15a^2 - 17b^2)x - 2a = 0$. Obtain the result without solving the given equations.

32. Solve the simultaneous equations:

$$\begin{cases} x^2 - y^2 = 1. \\ xy - 2y + x = 2. \end{cases}$$

33. There are four numbers of which the first three are in arithmetical progression and the last three in geometrical progression. Their sum is 21, and the third minus the first is 4. Find the numbers.

34. A farmer sold a horse at \$75 for which he had paid x dollars. He realized $x\%$ profit by his sale. Find x.

35. If the middle term of the expansion of $\left(3x - \frac{1}{2\sqrt{x}}\right)^4$ is equal to the fourth term of the expansion of $\left(2\sqrt{x} + \frac{1}{2x}\right)^7$, find the value of x.

36. Construct with respect to the same axes of reference the graphs of $y - x = 2$ and $y = x^2 - 4$. Solve these equations and show the relations between the roots and the graphs.

37. The difference between two numbers is 3, and the square of their geometric mean exceeds the arithmetic mean by $1\frac{1}{2}$. Find the numbers.

38. An employer hires a clerk for 5 years at a beginning salary of \$500 per year with either a rise of \$100 each year after the first or a rise of \$25 every 6 months after the first half year. Which is the better proposition for the clerk, and by how much?

39. Find two numbers x and y such that x, y, and xy are in geometric progression, and x, y, and $4x + 3$ are in arithmetic progression.

40. Solve for x: $\sqrt{3x + 10} = \sqrt{10x + 16} + \sqrt{x + 2}$. Which value of x satisfies the equation?

41. Solve for x: $6x^2 - 3x - 2 = \sqrt{2x^2 - x}$.

42. If a and b are the roots of the equation $hx^2 + px + q = 0$, find the equation whose roots are $(a + b)^2$ and $(a - b)^2$.

43. A farmer raised broom corn and pressed the 6120 pounds of brush into bales. If he had made each bale 20 pounds heavier, he would have had 1 bale less. How many bales did he press, and what was the weight of each?

44. Solve the simultaneous equations: $x^2 + 7xy + 12y^2 = 0$; $x^2 - (y - 2)^2 = 0$.

45. Solve the simultaneous equations:

$$\begin{cases} x^2 + y^2 = 24 + 5(x - y). \\ xy = 15. \end{cases}$$

46. Solve the simultaneous equations:

$$\frac{x^2}{y} = -12; \frac{x^2}{z^2} = 9; z = y + 1.$$

47. The diagonal of a rectangle is twice one side. Find the ratio of the sides.

48. The illumination from a source of light is inversely proportional to the square of the distance from the source. If my book is 8 feet from the lamp, how near must I move it in order that the light on it shall be doubled?

49. Find the value of .43131, . . . as a common fraction.

50. In the expansion of $\left(3x - \frac{1}{3x^{\frac{1}{2}}}\right)^7$ find the term which, when simplified, contains $x^{\frac{5}{2}}$.

51. In a mile race (1760 yards), A runs 6 yards per second and B 5 yards per second. B has a start of 250 yards. Take a horizontal line for the time axis and a vertical one for the distance axis and find graphically which one wins the race and how far he is in the lead at the finish.

52. The number 2 is a geometric mean between x and y, y is an arithmetic mean between $2x$ and z, and z exceeds the sum of $4x$ and y by 5. Find x, y, and z.

53. Find two numbers such that their sum, their difference, and the sum of their squares may be to each other as 4 : 1 : 17.

XXIV. LOGARITHMS

104.

1. Express the following in the form of logarithmic equations:

(*a*) $3^2 = 9$.

(*b*) $2^4 = 16$.

(*c*) $\sqrt[3]{8} = 2$.

(*d*) $5^{-3} = \frac{1}{125}$.

(*e*) $10^{-3} = .001$.

(*f*) $2^x = 32$.

(*g*) $4^{x+1} = 64$.

(*h*) $x^3 = 27$.

(*i*) $m^a = n$.

(*j*) $27^{\frac{4}{3}} = 81$.

2. Change the following to the form of exponential equations:

(*a*) $\log_b N = a$.

(*b*) $\log_3 9 = 2$.

(*c*) $\log_2 32 = 5$.

(*d*) $\log_{10} 1000 = 3$.

(*e*) $\log_{10} .0001 = -4$.

(*f*) $\log_{\frac{1}{3}} 81 = -2$.

(*g*) $\log_9 3 = \frac{1}{2}$.

(*h*) $\log_b 1 = 0$.

(*i*) $\log_x 8 = 3$.

(*j*) $\log_a b = c$.

3. When the base is 3, what are the logarithms of the following numbers: 9, $\frac{1}{3}$, 27, 1, $\frac{1}{81}$, 243?

4. When the base is 10, what are the logarithms of 1, $\frac{1}{10}$, 10, 100, .01, 1000, .001?

5. Solve for b, $\log_b \frac{1}{16} = -4$.

Solve the following equations:

6. $\log_2 8 = x$.

7. $\log_{10} \frac{1}{10} = x$.

8. $\log_{10} .001 = x$.

9. $3 \log_{27} 3 = x$.

10. $\log_4 8 = x$.

11. $\log_x 625 = 4$.

12. $\log_x \frac{1}{9} = -2$.

13. $\frac{1}{3} \log_3 27 = x$.

14. $\log_2 x = 5$.

15. $\log_5 2x = 2$.

105.

1. Apply the laws of logarithms to the following, thus expressing them in expanded form:

(*a*) $\log M \cdot N$.

(*b*) $\log \frac{M}{N}$.

(*c*) $\log M^p$.

(*d*) $\log \sqrt[a]{M}$.

(*e*) $\log 5cx^2$.

(*f*) $\log \frac{a^2}{b^3}$.

(*g*) $\log \sqrt{a^3 b^5}$.

(*h*) $\log \frac{a(a-b)}{a+b}$.

(*i*) $\log \frac{a^2 b^3 \sqrt{c}}{d^3 \sqrt[3]{e}}$.

(*j*) $\log P(1+r)^n$.

(*k*) $\log \frac{\sqrt[3]{100}}{(.01)^2}$.

(*j*) $\log \frac{(543)(78)^2}{\sqrt[3]{726}}$.

(*k*) $\log \frac{bc \sin A}{2}$.

2. Reversing the processes of the preceding, express the following in contracted form:

(*a*) $\log a - 2 \log b + \frac{1}{2} \log c$.

(*b*) $\log (a+b) + 2 \log (a-b)$.

(*c*) $\frac{1}{3} \log a^2 - 2 \log b^3$.

(*d*) $2 \log 895 + \frac{1}{2} \log 672 - \log 379$.

(*e*) $\frac{1}{2} [\log (s-a) + \log (s-b) + \log (s-c) + \log s]$.

3. Solve for x: $3 \log x = 2 \log 8$.

4. Given $\log_{10} 2 = .3010$, $\log_{10} 3 = .4771$, find $\log_2 3$, and $\log_2 60$.

5. If the base is 10 what is the characteristic, or integral part, of the logarithms of the following numbers: 182, .7, 3.1416, .0082, 24.3?

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
No.	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	0	1	2	3	4	5	6	7	8	9

106. Common Logarithms.

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 5 = .6990$, find:

1. $\log 6$.
2. $\log 15$.
3. $\log \frac{8}{2}$.
4. $\log \frac{5}{8}$.
5. $\log 2^3$.
6. $\log \frac{1}{2}^5$.
7. $\log \sqrt[3]{3}$.
8. $\log \sqrt{3}\ \sqrt[3]{5}$.
9. $\log 300$.
10. $\log .005^3$.
11. $\log 18$.
12. $\log \frac{.003^2}{.002^3}$.

Using the tables, find the logarithms of the following numbers:

13. 271.
14. 32.5.
15. 1.414.
16. .0278.
17. .548.
18. .00684.
19. 3.1416.
20. 253.4.

Using the tables, find the numbers corresponding to the following logarithms:

21. 1.8388.
22. $\bar{1}.6031$.
23. 2.7825.
24. 0.1492.
25. 1.7940.
26. $\bar{2}.3900$.
27. $9.5369 - 10$.
28. $8.7045 - 10$.

107. Computation with the Aid of Logarithms.

Find the value of the following, using logarithms:

1. 728×597.
2. $896 \div 65.3$.
3. $327 \times 238 \times .762$.
4. $(.0732) \div (-.00482)$.
5. $\frac{152 \times \sqrt{816}}{341}$.
6. $\sqrt[3]{892} \div \sqrt{76}$.
7. $298^{\frac{2}{3}}$.
8. $\frac{(.0048)(-1.87)}{(-.03271)}$.
9. $\frac{9.98^5 \times 123^2}{\sqrt[3]{158}}$.

10. $\frac{4}{3}\pi r^3$, if $\pi = 3.1416$ and $r = 30.7$.

The amount (A) which a given principal (P) will become at the end of n years when invested at compound interest at a given rate (r) is given by the formula

$$A = P(1 + r)^n$$

Using this formula solve the following problems with the aid of logarithms.

11. At the end of 8 years to what sum will \$2500 amount at 5%, if the interest is compounded annually?

12. At the birth of a child \$1000 is placed to his credit in a savings bank. If the bank pays interest at the rate of 4%, compounded annually, what will be the sum to his credit at his 21st birthday?

13. If the interest in problem 12 is compounded semi-annually what will be the sum at the 21st birthday?

14. What principal must be placed at $3\frac{1}{2}$%, compounded annually to yield an amount of \$1600 in 15 years?

15. What will be the amount of 20 cents placed at 5% for 300 years, the interest being compounded annually?

16. For how long must \$1400 be deposited at 5%, interest compounded annually, to yield \$20,000?

108. Examples in Logarithms from College Entrance Examinations.

1. Find the value of $\sqrt[3]{0.00034}$ by means of logarithms.

2. Compute the values of (*a*) $\sqrt{\frac{2.016 \times .06932}{.11264}}$; (*b*) $(1.01)^{25}$.

3. Calculate by logarithms

$$\left[\frac{2.186\sqrt{0.965}}{38.23}\right]^3$$

4. Find the value of $\log_2 \frac{1}{8^2} - \log_{10} \sqrt{.10}$.

5. Find by logarithms the value of $\sqrt[3]{\frac{0.6712}{5.327}}$.

6. Simplify: (*a*) $\log_2 4 + \log_3 9 + \log_{10} .1$.

(*b*) $\log_2 \frac{1}{2} + \log_3 \sqrt{9}$.

(*c*) $\log_{10} \sqrt{.001}$.

(*d*) $\log_3 10$.

7. Find by the logarithms the value of

$$\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \text{ where } \begin{aligned} a &= 8.341, \\ b &= 14.736, \\ c &= 7.499, \\ s &= \tfrac{1}{2}(a+b+c). \end{aligned}$$

8. Find by logarithms the fourth root of 0.0037956.

9. Find by logarithms the value of $e^{3.14}$, where $e = 2.72$.

10. Find, by means of logarithms, the value of:

(*a*) $\frac{3.765 \times 27.378}{(23.2)^5 \times (.26)^2}$;

(*b*) $\sqrt{.0371}$; $\sqrt[3]{17.45}$.

11. Using logarithms, compute the value of

$$\left[\frac{256.4\sqrt[3]{.07834}}{\sqrt{90.04}(8.634)^2}\right]^2.$$

12. Using logarithms compute the value of

(*a*) $\sqrt[4]{\frac{269.3}{8046}}$. (*b*) $5 - \sqrt[5]{21}$.

13. By the use of logarithms find the value of

$$\frac{(\sqrt{278.2} \times 2.578)^2}{\sqrt[3]{.00231} \times \sqrt{76.19}}.$$

14. Find by logarithms the value of

$$\frac{1}{\sqrt[3]{0.07343}}.$$

XXV. NUMERICAL TRIGONOMETRY

109. The Sine, Cosine, and Tangent of 45°, 30°, and 60°.

1. Construct a right triangle whose acute angles are 45°. From this figure, find the sine of 45°, cosine 45°, tangent 45°.

2. Construct a right triangle whose acute angles are 30° and 60° and find the sine of 30°, cosine of 30°, tangent of 30°; also sin 60°, cos 60°, tan 60°.

NOTE.—In the following problems, unless otherwise stated, the right triangles will be denoted by ABC, the acute angles by A and B, the right angle by C, and the lengths of the sides opposite these angles by the corresponding small letters, a, b, and c respectively.

3. In a right triangle, given $A = 30°$, $a = 40$ ft. Find b, c, and B.

4. In a right triangle, given $B = 60°$, $c = 15$ in. Find a and B.

5. The angle at the vertex of an isoceles triangle is 60°, and the equal sides are each 24 in. long. Find the base of the triangle and the area.

110. Problems Involving the Table of Trigonometric Ratios.

Solve the following right triangles, finding the remaining parts and the area:

1. $A = 36°$, $c = 1$.

2. $a = 10$, $B = 67°$.

3. $a = 36.4$, $b = 100$.

4. $A = 33°$, $c = 10.5$.

5. $a = 17.1$, $c = 50$.

6. $a = 48.04$, $A = 31° 45'$.

7. $b = 2.887$, $c = 5.11$.

8. $c = 627$, $A = 23° 30'$.

9. The base of an isosceles triangle is 68.4 feet and each of its equal sides is 100 feet. Find the angles and the altitude.

10. The altitude of an isoceles triangle is 10 in. and the vertex angle is 84°. Find the sides and area of the triangle.

11. The shadow of a tree is 42 ft. long when the angle of elevation of the sun is 65° 10′. Find the height of the tree.

12. Find the length of a side and the area of a regular pentagon inscribed in a circle whose radius is 12 feet.

13. If a pole 32 ft. high casts a shadow 40 ft. long on level ground what is the angle of elevation of the sun?

14. A chimney subtends from a certain point an angle of elevation of 30°; and from a point 250 ft. nearer an angle of 45°. What is the height of the chimney?

15. A chord 1 foot long subtends an angle of 1° at the center of a circle; find the distance of the chord from the center.

16. A man in a balloon, when it is one mile high, finds the angle of depression of an object on the level ground to be 35° 20′, then after ascending vertically some distance, he finds the angle of depression of the same object to be 55° 40′. What distance did he ascend between measurements?

17. Find the height of a chimney if the angle of elevation of its top changes from 22° to 37° when the observer walks toward it 140 ft. in a horizontal line through its base.

18. A car-wheel 40 inches in diameter has a flat place 5 inches long on the run. How much less is the perimeter of this wheel than that of a perfect wheel of the same diameter?

19. When an airship is 1050 ft. high above a level plane, find the distance of an object on the plane, the angle of depression being 72°.

TABLE OF TRIGONOMETRIC RATIOS

Angle	sin	cos	tan	Angle	sin	cos	tan
0°	.000	1.000	.000	**45°**	.707	.707	1.000
1°	.017	1.000	.017	46°	.719	.695	1.036
2°	.035	.999	.035	47°	.731	.682	1.072
3°	.052	.999	.052	48°	.743	.669	1.111
4°	.070	.998	.070	49°	.755	.656	1.150
5°	.087	.996	.087	**50°**	.766	.643	1.192
6°	.105	.995	.105	51°	.777	.629	1.235
7°	.122	.993	.123	52°	.788	.616	1.280
8°	.139	.990	.141	53°	.799	.602	1.327
9°	.156	.988	.158	54°	.809	.588	1.376
10°	.174	.985	.176	**55°**	.819	.574	1.428
11°	.191	.982	.194	56°	.829	.559	1.483
12°	.208	.978	.213	57°	.839	.545	1.540
13°	.225	.974	.231	58°	.848	.530	1.600
14°	.242	.970	.249	59°	.857	.515	1.664
15°	.259	.966	.268	**60°**	.866	.500	1.732
16°	.276	.961	.287	61°	.875	.485	1.804
17°	.292	.956	.306	62°	.883	.469	1.881
18°	.309	.951	.325	63°	.891	.454	1.963
19°	.326	.946	.344	64°	.899	.438	2.050
20°	.342	.940	.364	**65°**	.906	.423	2.145
21°	.358	.934	.384	66°	.914	.407	2.246
22°	.375	.927	.404	67°	.921	.391	2.356
23°	.391	.921	.424	68°	.927	.375	2.475
24°	.407	.914	.445	69°	.934	.358	2.605
25°	.423	.906	.466	**70°**	.940	.342	2.747
26°	.438	.899	.488	71°	.946	.326	2.904
27°	.454	.891	.510	72°	.951	.309	3.078
28°	.469	.883	.532	73°	.956	.292	3.271
29°	.485	.875	.554	74°	.961	.276	3.487
30°	.500	.866	.577	**75°**	.966	.259	3.732
31°	.515	.857	.601	76°	.970	.242	4.011
32°	.530	.848	.625	77°	.974	.225	4.331
33°	.545	.839	.649	78°	.978	.208	4.705
34°	.559	.829	.675	79°	.982	.191	5.145
35°	.574	.819	.700	**80°**	.985	.174	5.671
36°	.588	.809	.727	81°	.988	.156	6.314
37°	.602	.799	.754	82°	.990	.139	7.115
38°	.616	.788	.781	83°	.993	.122	8.144
39°	.629	.777	.810	84°	.995	.105	9.514
40°	.643	.766	.839	**85°**	.996	.087	11.430
41°	.656	.755	.869	86°	.9976	.070	14.301
42°	.669	.743	.900	87°	.9986	.052	19.081
43°	.682	.731	.933	88°	.9994	.035	28.636
44°	.695	.719	.966	89°	.9998	.017	57.290
45°	.707	.707	1.000	**90°**	1.000	.000	.....

EXAMINATION QUESTIONS

College Entrance Examination Board Examinations in Elementary Algebra

Mathematics A—Elementary Algebra Complete, 1921

Monday, June 20 9:30 a. m. Three hours

1. Factor:

(*a*) $20 - x - x^2$,

(*b*) $3x^3 - 81$,

(*c*) $a^4 - 13a^2 + 36$.

2. Solve the simultaneous equations:

$$2x - 5y - 10 = 0.$$
$$9x + 8y - 14.5 = 0.$$

3. Simplify:

$$\left(m + \frac{p^2}{m-p}\right) \div \left(m - \frac{(m-p)p^2}{m^2+p^2}\right).$$

4. Compute the positive root of the following equation to the nearest hundredth:

$$4x^2 - 9x - 11 = 0.$$

5. Given the formulas:

$$v = v_o - at, \qquad s = v_o t - \tfrac{1}{2}at^2.$$

Eliminate t and express v in terms of the other letters.

6. The federal income tax on incomes between \$12,000 and \$14,000 was in 1919 as follows: first, a uniform tax of \$190 on all such incomes; secondly, an additional tax of 5% on the excess of such an income over \$12,000. Write down a formula expressing the total tax, t, which a man must pay, whose income, x, lay between the foregoing limits.

7. (*a*) Find, and simplify, the fourth term in the binomial expansion of

$$\left(a - \frac{b}{\sqrt[3]{2}}\right)^{24}.$$

(*b*) Free the following expression from negative and fractional exponents, and from radicals:

$$\left(\sqrt[6]{5x^{\frac{7}{4}}y^{-\frac{2}{3}}}\right)^{12}.$$

8. Solve the simultaneous equations:

$$xy = 8, \qquad 2x + y = 10.$$

9. Plot the graphs of the equations of Question **8,** and thus check your answers.

10. A workman, wishing to explode a blast of powder, set the fuse to cause the explosion to take place in 30 seconds. He ran back at the rate of eight yards per second. How far had he run when he heard the explosion, if sound travels at the rate of 1,080 feet per second?

Mathematics A—Elementary Algebra Complete 1920

Monday, June 21 9:30 a.m. Three hours

1. Factor

$$35 + 11x - 6x^2,$$
$$x^2 - a^2 + y^2 - b^2 - 2xy + 2ab.$$

2. Express as a fraction in its lowest terms

$$\frac{1}{ax}\left(\frac{a}{x} - \frac{x}{a}\right) \div \frac{a^3 + x^3}{a^3x^3}.$$

3. Solve the equation

$$3x^2 - 4x - 5 = 0,$$

computing the roots to the nearest hundredth.

4. Solve the equation

$$\frac{x-1}{x-2} - 2 - \frac{x}{1-x} = 0.$$

Test your answer.

5. The sides of a rectangle are in the ratio 5:2. If 2 inches are added to each side, the ratio is 4:3. Find the sides.

6. Solve the simultaneous equations:

$$y = x^2 - 4x + 3,$$
$$4x + 4y - 7 = 0.$$

Pair the results clearly.

7. Plot the graphs of the equations of question 6, and mark the values of x and y at the points of intersection.

8. Simplify

$$\left(x^{\frac{3}{2}} - y^{-\frac{3}{2}}\right)^2 + 2\left(\sqrt{\frac{x}{y}}\right)^3.$$

9. A man wishing to travel from a town A to a town B, 65 miles away, finds it convenient to go by automobile from A to another town C and thence by trolley to B. The triangle ABC is assumed to have a right angle at C. If the automobile averages 20 miles an hour and the trolley 10 miles an hour, and the total time of the trip, exclusive of the time spent at C, is $5\frac{1}{2}$ hours, find the distances AC and CB.

NOTE.—Make use of the fact that in the triangle ABC the square of the side opposite the right angle is equal to the sum of the squares of the other two sides.

10. A car slipping backward down hill moves 2 inches in the first second, 6 inches in the second second, 10 inches in the third second and so on. How far will it move in 20 seconds?

Mathematics A—Elementary Algebra Complete, 1919

Monday, June 16 9:30 a.m. Three hours

1. (*a*) Factor

$$3mx^2 - 8mx - 3m,$$
$$x^4 + 64x,$$
$$1 - 2ax - (c - a^2)x^2 + acx^3.$$

(*b*) Simplify

$$\frac{\frac{1}{a} - \frac{b+a}{b^2+a^2}}{\frac{1}{b} - \frac{b+a}{b^2+a^2}},$$

and test the result by setting $a = 1$ and $b = 2$ in the given expression and in your result.

2. (*a*) Find the value of each of the following expressions:

$$\frac{(x+y)^0}{2^{-3}}, \quad (4)^{\frac{3}{2}}, \quad \frac{1^{3x}}{2^{-1} - 3^{-1}}.$$

(*b*) Given $x^4(x^2y - xyz)^{-2}$. Re-write the expression, without changing its value, so that x appears only within the parenthesis and y only outside the parenthesis.

3. (*a*) Solve $3x^2 + 2x = 11$. Give the values of x correct to the nearest hundredth.

(*b*) Find the relation p and q, for which one root of $x^2 + px + q = 0$ is three times the other.

4. (*a*) Find the value of

$$\sqrt{13 + \sqrt{2} + \frac{7}{3 + \sqrt{2}}}$$

(*b*) Solve $x^2 - 2\sqrt{x^2 + 16} = -1$

Substitute your results in the equation and explain the failure of some of them to satisfy the equation.

5. (*a*) Solve

$$\frac{1}{x} + \frac{1}{y} = 5,$$
$$x - y = 0.3.$$

(*b*) Write, and simplify, that term of the expansion of $\left(\frac{x}{\sqrt{y}} - y^{\frac{3}{4}}\right)^8$ which has the greatest coefficient.

6. (*a*) Solve

$$y = 3x - 12,$$
$$x^2 - y^2 = 16.$$

(*b*) Plot the curves represented by the two equations in 6(*a*) and check the solution of 6(*a*) from the graph.

7. (*a*) Write down, and prove, the formula for the sum of n terms of the series a, $a + d$, $a + 2d$, etc.

(*b*) A farmer sowed $\frac{1}{8}$ bushel of wheat and used the whole produce for seed the following year, the produce of this second year for seed the third year, and the produce of the third year for seed the fourth year. The fourth harvest provided 12,800 bushels. Assuming the rate of increase to have been the same for each of the four years, how many bushels did the second harvest yield?

8. An automobile makes a trip of 150 miles at a constant speed. Returning, it travels $2\frac{1}{2}$ miles an hour faster and returns in 40 minutes' less time than it took to go out. Required, the speed of the car on the outward trip.

Mathematics A—Elementary Algebra Complete, 1918

Monday, June 17 9:30 a.m. Three hours

1. (*a*) Factor

$$3x^2 + 5x - 12,$$
$$(a^2 - ab)^2 - (ab - b^2)^2,$$
$$x^2 + y^2 + abx - (2x + ab)y.$$

(*b*) Simplify

$$\frac{\frac{1}{x}-\frac{y}{x^2}}{1+\frac{y^2}{x^2}} \div \frac{\frac{1}{x}+\frac{y}{x^2}}{1-\frac{y^2}{x^2}}.$$

2. Solve the following simultaneous equations and verify the solution:

$$\begin{aligned} x + y + z &= -2, \\ 3x + 2z &= 0, \\ 5y - 3z &= 4. \end{aligned}$$

3. (*a*) Simplify the expression $\sqrt{28}+\frac{1}{8-3\sqrt{7}}$ and calculate its value correct to the nearest hundredth.

(*b*) Find, and express in its simplest form, the fourth term of $(\sqrt[4]{m} - \sqrt[3]{5})^{11}$.

4. (*a*) Solve $x+\frac{1}{x-1}=\frac{9}{2}$.

(*b*) Solve

$$(x^2 - 1)^2 - 11(x^2 - 1) + 24 = 0.$$

5. (*a*) Solve the simultaneous equations:

$$\begin{aligned} y &= x^2 + 2x - 5, \\ 3x &= y + 3. \end{aligned}$$

(*b*) Plot the graphs of these two equations on the same set of axes and show how the figure checks the solutions found in (*a*).

6. Solve $\sqrt{2x+1}+\sqrt{x-3}=\sqrt{5x+4}$. Show whether your results satisfy the given equation.

Mathematics A—Elementary Algebra Complete 1917

Monday . 9:30 a.m. Three hours

1. (*a*) Factor

$$6x^2 - x - 77,$$
$$\frac{x^2}{y^2} + \frac{y^2}{x^2} + 2,$$
$$a^3 + b^3 + a + b.$$

(*b*) Simplify

$$3x^2 - [7x - 2 - (2x - 1)(3 - x)].$$

2. (*a*) Simplify

$$\left[\frac{2}{x} - \frac{1}{a+x} + \frac{1}{a-x}\right] \div \left[\frac{a+x}{a-x} - \frac{a-x}{a+x}\right].$$

(*b*) Solve, and verify your answer,

$$\frac{4(x+3)}{9} = \frac{8x+37}{18} - \frac{7x-29}{5x-12}.$$

3. (*a*) Find, and express in its simplest form, the sixth term of $\left(\frac{x}{2} - \frac{4}{x^2}\right)^8$.

(*b*) Divide $2a - a^{\frac{1}{2}}b^{-\frac{1}{2}} - 3b^{-1}$ by $2a^{\frac{1}{2}} - 3b^{-\frac{1}{2}}$, and verify the result by placing $a = 9$ and $b = 4$ in dividend, divisor, and quotient.

4. (*a*) Solve $.5x^2 + 1.5x + .88 = 0$.

(*b*) Find the values of k for which $x^2 - 6kx + 12 = 0$ has equal roots.

5. Solve, and verify the positive answer,

$$\sqrt{3x+1} + \frac{35}{\sqrt{3x+1}} = 3\sqrt{x}.$$

6. If A gives B \$6, B will have $\frac{2}{3}$ as much as A has left; but if B gives A \$5, B will have $\frac{3}{5}$ as much as A then has. How much had each?

7. The sum of three terms in arithmetical progression beginning with $\frac{8}{3}$ is equal to the sum of three terms of a geometrical progression beginning with $\frac{8}{3}$, and the common difference is equal to the ratio. What are the two series?

8. A man bought a certain number of oranges for \$2. If he had paid 5 cents more per dozen, he would have received two dozen less for the same money. How much did he pay for a dozen?

Mathematics A—Elementary Algebra Complete, 1916

Monday 9:00 a.m. Two hours

1. (*a*) Factor

$$6x^2 - 5x - 4,$$
$$30m + 4p^2 - 25 - 9m^2,$$
$$x(x + 2)(2x - 1) - (x + 2).$$

(*b*) Find the value of

$$\frac{x - a}{x}\sqrt{x^2 - 4},$$

when $x = a + \frac{1}{a}$.

2. (*a*) What are the values of

$$(a + b)^0,\ (8)^{-\frac{2}{3}},\ \frac{1}{2^{-1} + 3^{-1}}.$$

(*b*) Simplify

$$\frac{\sqrt{5 - 2\sqrt{6}} + \sqrt{5 + \sqrt{24}}}{3 - \sqrt{5}}.$$

3. Solve

$$\frac{x + 1}{2x^2 + 3x - 2} + \frac{2 - x}{1 - x - 2x^2} + \frac{x - 2}{x^2 + 3x + 2} = 0.$$

4. Plot the equations $xy = 4$ and $2x - 3y = 5$ on the same axes, and from the figure estimate the solutions of the equations.

5. Solve

$$\begin{cases} x^2 - xy = 2x + 5, \\ x^2 - xy = 3y + 9. \end{cases}$$

6. The difference between two numbers is 3, and the square of their geometric mean exceeds 75% of their arithmetic mean by $\frac{1}{4}$. Find the numbers.

7. Two bodies, one starting from A and the other from B at the same time, move toward each other and meet in $17\frac{1}{2}$ sec. If it takes one 12 sec. longer than the other to move between A and B, and the rate of the faster is 7 ft. per sec., how long does it take each to traverse the distance?